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1 (Sem-4) MAT 4

2025

**MATHEMATICS**

Paper : MAT0400404

**(Number Theory-I)**

Full Marks : 60

Time : 2½ hours

**The figures in the margin indicate  
full marks for the questions.**

1. Answer the following questions : 1×8=8

(a) State Well-Ordering Principle.

(b) If  $a$  and  $b$  are integers with  $b \neq 0$ , then there exist unique integers  $q$  and  $r$  such that  $a = qb + r$  where

(i)  $0 < r \leq b$

(ii)  $0 \leq r < |b|$

(iii)  $0 \leq r \leq b$

(iv)  $0 \leq r \leq |b|$



(Choose the correct option)



(c) Which of the following Diophantine equation cannot be solved ?

(i)  $6x + 51y = 22$

(ii)  $24x + 138y = 18$

(iii)  $158x - 57y = 7$

(iv)  $221x + 35y = 11$

(d) Give an example to show that  $a^2 \equiv b^2 \pmod{n}$  need not imply that  $a \equiv b \pmod{n}$ .

(e) Without performing the division, determine whether the integer 176, 521, 221, is divisible by 9.

(f) If  $p$  is a prime number, then

(i)  $(p-1)! \equiv 1 \pmod{p}$

(ii)  $(p-1)! \equiv -1 \pmod{p}$

(iii)  $(p+1)! \equiv 1 \pmod{p}$

(iv)  $(p+1)! \equiv -1 \pmod{p}$

(Choose the correct option)

(g) Find  $\sigma(180)$ .

(h) Define Möbius  $\mu$ -function.

2. Answer the following questions :  $2 \times 6 = 12$

(a) If  $a|c$  and  $b|c$  with  $\gcd(a,b)=1$ , then prove that  $ab|c$ .

(b) Prove that  $\gcd(a+b, a-b)=1$  or 2 if  $\gcd(a,b)=1$ .

(c) Use Fermat's theorem to show that  $5^{38} \equiv 4 \pmod{11}$ .

(d) Show that 41 divides  $2^{20} - 1$ .

(e) If  $n$  is a square free integer, prove that  $\tau(n) = 2^r$ , where  $r$  is the number of prime divisors of  $n$ .

(f) For  $n > 2$ , prove that  $\phi(n)$  is an even integer.



3: Answer **any four** of the following questions :

$$5 \times 4 = 20$$

(a) State and prove Archimedean property.

$$1 + 4 = 5$$

(b) Use the Euclidean Algorithm to obtain integers  $x$  and  $y$  satisfying

$$\gcd(12378, 3054) = 12378x + 3054y$$

(c) Use Chinese Remainder Theorem to solve the simultaneous congruences

$$x \equiv 2 \pmod{3}$$

$$x \equiv 3 \pmod{5}$$

$$x \equiv 2 \pmod{7}$$

(d) If  $n$  and  $r$  are positive integers with  $1 \leq r < n$ , then prove that the binomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is also an integer.

(e) Prove that every positive integer  $n > 1$  can be expressed uniquely as a product of primes a part from the order in which the factors occur.

(f) If  $p$  and  $q$  are distinct primes, prove that  $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$

(g) If  $f$  is a multiplicative function and  $F$

be defined by  $F(n) = \sum_{d|n} f(d)$ , then

prove that  $F$  is also multiplicative.

(h) If  $n \geq 1$  and  $\gcd(a, n) = 1$ , then prove that

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

4. Answer **any two** of the following questions :  $10 \times 2 = 20$

(a) (i) Prove that for given integers  $a$  and  $b$ , with  $b > 0$ , there exist unique integers  $q$  and  $r$  satisfying

$$a = bq + r, \quad 0 \leq r < b$$

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(ii) Establish the following formula by Mathematical induction.

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all  $n \geq 1$

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- (b) (i) Given integers  $a$  and  $b$ , not both of which are zero, prove that there exist integers  $x$  and  $y$  such that  $\gcd(a, b) = ax + by$ . 5

- (ii) Determine all solutions in the positive integers of the Diophantine equation

$$172x + 20y = 1000 \quad 5$$

- (c) (i) Prove that if  $p$  is a prime and  $p \nmid a$ , then

$$a^{p-1} \equiv 1 \pmod{p}$$

Is the converse of it true? Justify.

$$5+1=6$$

- (ii) Solve:  $9x \equiv 21 \pmod{30}$ . 4

- (d) (i) Prove that there is an infinite number of primes. 4

- (ii) Prove that the quadratic congruence  $x^2 + 1 \equiv 0 \pmod{p}$  where  $p$  is an odd prime has a solution if and only if  $p \equiv 1 \pmod{4}$ . 6

- (e) (i) If  $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$  is the prime factorization of  $n > 1$ , then prove that

$$(I) \quad \tau(n) = (k_1 + 1)(k_2 + 1) \dots (k_r + 1)$$

$$(III) \quad \sigma(n) = \left( \frac{p_1^{k_1+1} - 1}{p_1 - 1} \right) \left( \frac{p_2^{k_2+1} - 1}{p_2 - 1} \right) \dots \left( \frac{p_r^{k_r+1} - 1}{p_r - 1} \right)$$

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- (ii) For each positive integer  $n \geq 1$ , prove that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$

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