1 (Sem-4) MAT 4

2025

MATHEMATICS

Paper: MAT0400404

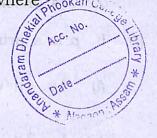
(Number Theory-I)

Full Marks: 60

Time: 21/2 hours

The figures in the margin indicate full marks for the questions.

- 1. Answer the following questions: $1 \times 8 = 8$
 - (a) State Well-Ordering Principle.
 - (b) If a and b are integers with $b\neq 0$, then there exist unique integers q and r such that a = qb + r where one of an are integers with a = ab + r where one of a such that a = ab + r where
 - (i) $0 < r \le b$
 - (ii) $0 \le r < |b|$
 - (iii) $0 \le r \le b$
 - (iv) $0 \le r \le |b|$



(Choose the correct option)

(c) Which of the following Diophantine equation cannot be solved?

(i)
$$6x + 51y = 22\sqrt{5}$$

(ii)
$$24x + 138y = 18$$

(iii)
$$158x - 57y = 7$$

(iv)
$$221x + 35y = 11$$

- (d) Give an example to show that $a^2 \equiv b^2 \pmod{n}$ need not imply that $a \equiv b \pmod{n}$.
- (e) Without performing the division, determine whether the integer 176, 521, 221, is divisible by 9.
- (f) If p is a prime number, then

(i)
$$(p-1)! \equiv 1 \pmod{p}$$

(ii)
$$(p-1)! \equiv -1 \pmod{p}$$

(iii)
$$(p+1)! \equiv 1 \pmod{p}$$

(iv)
$$(p+1)! \equiv -1 \pmod{p}$$

(Choose the correct option)

- (g) Find $\sigma(180)$.
- (h) Define Möbius μ -function.
- 2. Answer the following questions: 2×6=12
 - (a) If $a \mid c$ and $b \mid c$ with gcd (a,b)=1, then prove that $ab \mid c$.
 - (b) Prove that gcd(a+b,a-b)=1 or 2 if gcd(a,b)=1.
 - (c) Use Fermat's theorem to show that $5^{38} \equiv 4 \pmod{11}$.
 - (d) Show that 41 divides $2^{20} 1$.
 - (e) If n is a square free integer, prove that $\tau(n) = 2^r$, where r is the number of prime divisors of n.
 - (f) For n>2, prove that $\phi(n)$ is an even integer.

- 3: Answer any four of the following questions: 5×4=20
 - State and prove Archimedean property. 1+4=5
 - (b) Use the Euclidean Algorithm to obtain integers x and y satisfying gcd(12378, 3054) = 12378x + 3054y
 - Use Chinese Remainder Theorem to solve the simultaneous congruences

$$x \equiv 2 \pmod{3}$$

$$x \equiv 3 \pmod{5}$$

$$x \equiv 2 \pmod{7}$$

If n and r are positive integers with $1 \le r < n$, then prove that the bionomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is also an integer.

(e) Prove that every positive integer n>1 can be expressed uniquely as a product of primes a part from the order in which the factors occur.

- If p and q are distinct primes, prove that $p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}$
- (g) If f is a multiplicative function and Fbe defined by $F(n) = \sum_{d|n} f(d)$, then

prove that F is also multiplicative.

(h) If $n \ge 1$ and gcd(a,n)=1, then prove that $a^{\phi(n)} \equiv 1 \pmod{n}$.

4. Answer **any two** of the following questions:

- 10×2=20
 - Prove that for given integers a and b, with b>0, there exist unique integers q and r satisfying 6 a = bq + r, $0 \le r < b$
 - Establish the following formula by Mathematical induction.

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$
for all $n \ge 1$

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- (b) (i) Given integers a and b, not both of which are zero, prove that there exist integers x and y such that gcd(a,b) = ax + by.
 - (ii) Determine all solutions in the positive integers of the Diophantine equation 172x + 20y = 1000
- Prove that if p is a prime and $p \nmid a$,

 then $a^{p} \equiv 1 \pmod{p}$
 - Is the converse of it true? Justify. 5+1=6
 - (ii) Solve: $9x = 21 \pmod{30}$.
 - (d) (i) Prove that there is an infinite number of primes.
 - (ii) Prove that the quadratic congruence $x^2 + 1 \equiv 0 \pmod{p}$ where p is an odd prime has a solution if and only if $p \equiv 1 \pmod{4}$.

- (e) (i) If $n = p_1^{k_1} p_2^{k_2} p_r^{k_r}$ is the prime factorization of n > 1, then prove that
- (I) $\tau(n) = (k_1 + 1)(k_2 + 1)...(k_\tau + 1)$

(II)
$$\sigma(n) = \left(\frac{p_1^{k_1+1}-1}{p_1-1}\right) \left(\frac{p_2^{k_2+1}-1}{p_2-1}\right) \cdots \left(\frac{p_r^{k_r+1}-1}{p_r-1}\right)$$

For each positive integer $n \ge 1$,

prove that $\begin{pmatrix}
1 & \text{if } n = 1
\end{pmatrix}$

$$\mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases}$$

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