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3 (Sem–4/CBCS) MAT HC 3

2025

MATHEMATICS

(Honours Core)

Paper : MAT–HC–4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

(a) Define characteristic of a ring.

(b) State whether the following statement is *True or False* :
"$2\mathbb{Z}\ U\ 3\mathbb{Z}$ is a subring of $\mathbb{Z}$".

(c) In the ring of integers, find a positive integer x such that $\langle x\rangle=\langle m\rangle+\langle n\rangle$.

(d) How many zeros does the equation $x^2 + 3x + 2 = 0$ have in $\mathbb{Z}_6$?

(e) Find all the maximal ideals in $\mathbb{Z}_8$.

(f) What is the characteristic of $\mathbb{Z}_m \oplus \mathbb{Z}_n$?

(g) Is the polynomial $x^2 + x + 4$ irreducible over $\mathbb{Z}_{11}$?

(h) State whether the following statement is *True* **or** *False* :
"$\mathbb{Z}_6$ is a subring of $\mathbb{Z}_{12}$".

(i) Define prime ideal of a ring.

(j) Given an example of a UFD, which is not a PID.

2. Answer the following questions : 2×5=10

(a) Show that the centre of a ring is a subring.

(b) Prove that the only ideals of a field $\mathbb{F}$ are $\{o\}$ and $\mathbb{F}$ itself.

(c) Is the mapping $\varphi : \mathbb{Z}_5 \to \mathbb{Z}_{30}$ given by $\phi(x) = 6x$ a ring homomorphism ?

(d) Consider $f(x) = x^3 + 2x + 4$ and $g(x) = 3x + 2$ in $\mathbb{Z}_5[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

(e) Let $f(x) = x^3 + x^2 + x + 1 \in \mathbb{Z}_2[x]$. Write $f(x)$ as a product of irreducible polynomials over $\mathbb{Z}_2$.

3. Answer **any four** questions : 5×4=20

(a) Let x be a positive integer. Show that $Q\left[\sqrt{x}\right] = \{a + b\sqrt{x} : a, b \in Q\}$ is a field.

(b) Let R be a commutative ring with unity. Show that an ideal A of R is prime if and only if the quotient ring R/A is an integral domain.

(c) Define integral domain. Prove that if D is an integral domain, then the polynomial ring $D[x]$ is also an integral domain. 1+4=5

(d) Show that every Euclidean domain is a principal ideal domain.

(e) Show that x^4+1 is irreducible over Q but reducible over $\mathbb{R}$.

(f) Let D be a PID and let $p \in D$. Prove that $\langle p \rangle$ is maximal in D if and only if p is irreducible.

4. Answer the following questions : 10×4=40

(a) (i) Define subring. Prove that a nonempty subset S of a ring R is a subring. If S is closed under subtraction and multiplication. 1+5=6

(ii) Let $R = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $S = \{(a,b,c) \in R : a+b=c\}$. Prove or disprove that S is a subring of R. 4

Or

(i) Let R be a finite commutative ring with unity. Prove that every nonzero element of R is either a zero-divisor or a unit. 5

(ii) Describe all zero-divisors and units of $\mathbb{Z} \oplus Q \oplus \mathbb{Z}$. 5

(b) (i) Let ϕ be a ring homomorphism from R to S. Then the mapping from $R/Ker\,\phi$ to $\phi(R)$, given by $r + Ker\,\phi \to \phi(r)$ is an isomorphism, i.e., $R/Ker\,\phi \cong \phi(R)$. 6

(ii) Let $S = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a,b \in \mathbb{R} \right\}$. Show that $\phi : \mathbb{C} \to S$ given by $\phi(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a ring isomorphism. 4

OR

For a field $\mathbb{F}$, define and prove the division algorithm for $\mathbb{F}[x]$. 2+8=10

(c) (i) Prove that $\mathbb{Z}\left[\sqrt{5}\right]$ is not a unique factorization domain. 4

(ii) Show that the ring of Gaussian integers $\mathbb{Z}[i]=\{a+ib : a,b\in\mathbb{Z}\}$ is a Euclidean domain with $d(a+ib)=a^2+b^2$. 6

OR

(i) Find all units, zero-divisors, idempotents and nilpotent elements in $\mathbb{Z}_3\times\mathbb{Z}_6$. 6

(ii) Let $\mathbb{F}$ be a field of prime characteristic p. Prove that $K=\{x\in\mathbb{F} : x^p=x\}$ is a sub field of $\mathbb{F}$. 4

(d) (i) In $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients, let $I=\{f(x)\in\mathbb{Z}[x] : f(0)=0\}$. Prove that $I=\langle x\rangle$. 5

(ii) Show that $\mathbb{R}/\langle x^2+1\rangle$ is a field. 3

(iii) Show that the kernel of a homomorphism is an ideal. 2

OR

(i) Let $\mathbb{F}$ be a field, then show that $\mathbb{F}[x]$ is a principal ideal domain. 5

(ii) Let

$f(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots\ldots+a_0\in\mathbb{Z}(x)$

If there is a prime p such that $p\nmid a_n$, $p|a_{n-1},\ldots.\, p|a_0$ and $p^2\nmid a_0$. Then show that $f(x)$ is irreducible over Q. 5
