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3 (Sem–2/CBCS) MAT HC 1

2025

MATHEMATICS

(Honours Core)

Paper : MAT–HC–2016

(Real Analysis)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

(a) If $S=\left\{\frac{1}{n}:n\in N\right\}$, then find the value of inf S.

(b) "The unit interval $[0,1]$ is uncountable." *(State True or False)*

(c) State Bolzano's Intermediate Value theorem.

(d) Give an example of a bounded sequence that is not a Cauchy sequence.

(e) "Every bounded sequence with a unique limit point is convergent." *(State True or False)*

(f) Give an example of a set which is not bounded above.

(g) For what value of ℓ, $\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right)=\ell$ converges where $\sum u_n$ is a positive term series.

(h) Define an alternating series.

(i) "Every absolutely convergent series is convergent." *(State True or False)*

(j) What is the value of $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

2. Answer the following questions : 2×5=10

(a) If $S=\{1-(-1)^n/n : n\in N\}$, find inf S.

(b) Prove that $\lim\left(1/(n^2+1)\right)=0$.

(c) Examine the sequence $\left(1, \frac{1}{2}, 3, \frac{1}{4}, \ldots\right)$ is convergent or divergent.

(d) Define Cauchy sequence.

(e) If $a, b \in R$, then show that $a^2+b^2=0$ if and only if $a=0$ and $b=0$.

3. Answer ***any four*** questions : 5×4=20

(a) Show that the set S of real number is bounded iff there exists a real number $G>0$ such that $|x|\le G, \forall x\in S$.

(b) Prove that if $I_n=[a_n, b_n], n\in N$ is a nested sequence of closed, bounded intervals such that the lengths b_n-a_n of I_n satisfy inf $\{b_n-a_n : n\in N\}=0$, then the number ξ contained in I_n for all $n\in N$.

(c) Prove that a sequence in R can have at most one limit.

(d) Prove that if $X=(x_n)$ is a convergent sequence of real numbers and if $x_n\ge 0$, for all $n\in N$, then $x=\lim(x_n)\ge 0$.

Contd.

(e) Examine the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is not convergent.

(f) Prove that if $\sum u_n$ is a positive term series, such that $\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \ell$, then the series

(i) converges, if $\ell < 1$;

(ii) diverges, if $\ell > 1$.

4. Answer the following questions : 10×4=40

(a) State and Prove Bolzano-Weierstrass Theorem.

Or

Test for convergence of the series whose nth term is $\left\{(n^2+1)^{\frac{1}{3}} - n\right\}$.

(b) Show that $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \le 1$.

Or

(i) Show that the series

$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \ldots + \left(\frac{1}{(\log n)^p}\right)$$

diverges for $p > 0$. 5

(ii) For real number $x, a, \varepsilon > 0$, show that

(i) $|x| < \varepsilon \Leftrightarrow -\varepsilon < x < \varepsilon$

(ii) $|x-a| < \varepsilon \Leftrightarrow a-\varepsilon < x < a+\varepsilon$

5

(c) If $\{a_n\}$ and $\{b_n\}$ are bounded sequences, then

(i) $\underline{\lim}\, a_n + \underline{\lim}\, b_n \le \underline{\lim}\,(a_n + b_n)$

(ii) $\underline{\lim}\, a_n + \overline{\lim}\, b_n \le \overline{\lim}\,(a_n + b_n)$

Or

(i) Test the convergence of the series $\sum \frac{1}{n^{1+1/n}}$ 5

(ii) Test the convergence of the series

$$\frac{\alpha}{\beta}+\frac{(1+\alpha)}{(1+\beta)}+\frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)}+\ldots.. \quad 5$$

(d) (i)

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots.$$

5

(ii) Show that the sequence

$$a_{n+1}=\frac{1}{2}\left(a_n+\frac{9}{a_n}\right), n\geq 1 \text{ and } a_1>0$$

converges to 3. 5

Or

(i) Show that for any fixed value of x the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ is convergent. 5

(ii) Show that

$$\lim_{n\to\infty}\frac{m(m-1)\ldots..(m-n+1)}{(n-1)!}x^n=0,$$

where $|x|<1$ and m is any real number. 5