Total number of printed pages–7

3 (Sem–5/CBCS) MAT HC 1

2024

MATHEMATICS

(Honours Core)

(New Course)

Paper : MAT–HC–5016

(Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×7=7

   (a) The function

   $f(z) = xy^2 + e^{xy} + i(2x - y)$ is

   continuous everywhere in the complex plane. *(State True* ***or*** *False)*

Contd.

(b) Let the function $f(z) = u(x,y) + iv(x,y)$ be defined throughout some ε-neighbourhood of a point $z_0 = x_0 + iy_0$. State a sufficient condition for existence of the derivative $f'(z_0)$.

(c) Find $\lim_{z \to -1} \frac{iz+3}{z+1}$.

(d) Define entire function and give an example.

(e) Show that $exp(2 \pm 3\pi i) = -e^2$.

(f) What is a Jordan curve?

(g) State Liouville's theorem.

2. Answer the following questions : 2×4=8

(a) Using $\varepsilon - \delta$ definition, show that if $f(z) = z^2$ then $\lim_{z \to z_0} f(z) = z_0^2$.

(b) Show that $f(z) = \begin{Bmatrix} z^2, & z \neq z_0 \\ 0, & z = z_0 \end{Bmatrix}$ where $z_0 \neq 0$ is discontinuous at $z = z_0$.

(c) Determine the singular points of the function, $f(z) = \frac{2z+1}{z(z^2+1)}$.

(d) Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by $z = t^2 + it$.

3. Answer **any three** questions from the following : 5×3=15

(a) Show that the three cube roots of $-8i$ lie at the vertices of an equilateral triangle that is inscribed in a circle of radius 2 centred at the origin.

(b) Prove that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

(c) If $f'(z) = 0$ everywhere in a domain D, then prove that $f(z)$ must be constant throughout D.

(d) Suppose that a function $f(z)$ is analytic at a point $z_0 = z(t_0)$ on a differentiable arc $z = z(t)(a \le t \le b)$. Show that if $w(t) = f(z(t))$ then $w'(t) = f'(z(t))z'(t)$ when $t = t_0$.

(e) Using anti-derivative, evaluate the integral $\int_C z^{1/2} dz$, where C is a contour from $z=-3$ to $z=3$ that, except for its end points, lies above the x-axis.

4. Answer ***any three*** questions from the following : 10×3=30

(a) (i) If z_0 and w_0 are points in the z and w planes respectively, then prove that

(A) $\lim_{z \to z_0} f(z) = \infty$ if and only if

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0 \text{ and}$$

(B) $\lim_{z \to \infty} f(z) = w_0$ if and only if

$$\lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$ 5

(ii) Let u and v denote the real and imaginary components of the function f defined by the equations :

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & \text{when } z \neq 0 \\ 0, & \text{when } z = 0 \end{cases}$$

Verify the Cauchy-Riemann equations at the origin $z=(0,0)$. 5

(b) (i) Show that

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y.$$ 2

(ii) Find numbers $z = x+iy$ such that $e^z = 1+i$. 3

(iii) Show that if a function $f(z) = u(x,y) + iv(x,y)$ and its conjugate $\overline{f(z)}$ are both analytic in a domain D, then $f(z)$ must be constant throughout D. 5

(c) (i) Show that the zeros of $\sin z$ are all real. 2

(ii) Evaluate $\int_0^{\pi/4} e^{it}\, dt$. 3

(iii) If $w = f(z) = \frac{1+z}{1-z}$ find $\frac{dw}{dz}$ and determine where $f(z)$ is not analytic. 5

Contd.

(d) (i) If $w(t)$ is a piecewise continuous complex valued function defined on an interval $a \le t \le b$, then show that $\left|\int_a^b w(t)\,dt\right| \le \int_a^b |w(t)|\,dt$ 5

(ii) Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that $\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$, without evaluating the integral. 5

(e) (i) Show that

$$\int \frac{dz}{z^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{z}{a} + C_1 = \frac{1}{2ai}\ln\left(\frac{z - ai}{z + ai}\right) + C_2$$

5

(ii) Evaluate :

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz,$$ 5

where C is the circle $|z| = 3$.

(f) (i) Prove that if a function f is analytic at a given point, then its derivatives of all orders are also analytic at that point. 5

(ii) Let C denote the positively oriented boundary of a square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate

$$\int_C \frac{\tan(z/2)}{(z - x_0)^2}\,dz \quad (-2 < x_0 < 2).$$ 5
