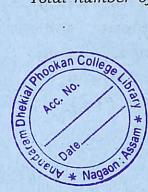
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3 (Sem-5/CBCS) MAT HC 1



2024

MATHEMATICS

(Honours Core)

(New Course)

Paper: MAT-HC-5016

(Complex Analysis)

Full Marks: 60

Time: Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions: 1×7=7

(a) The function

$$f(z) = xy^2 + e^{xy} + i(2x - y)$$
 is

continuous everywhere in the complex plane. (State True or False)

- (b) Let the function f(z) = u(x,y) + iv(x,y)be defined throughout some ε -neighbourhood of a point $z_0 = x_0 + iy_0$. State a sufficient condition for existence of the derivative $f'(z_0)$.
- (c) Find $\lim_{z \to -1} \frac{iz+3}{z+1}$.
- (d) Define entire function and give an example.
- (e) Show that $exp(2\pm 3\pi i) = -e^2$.
- (f) What is a Jordan curve?
- (g) State Liouville's theorem.
- 2. Answer the following questions: $2\times4=8$
 - (a) Using $\varepsilon \delta$ definition, show that if $f(z) = z^2$ then $\lim_{z \to z_0} f(z) = z_0^2$.
 - (b) Show that $f(z) = \begin{cases} z^2, z \neq z_0 \\ 0, z = z_0 \end{cases}$ where $z_0 \neq 0$ is discontinuous at $z = z_0$.

- (c) Determine the singular points of the function, $f(z) = \frac{2z+1}{z(z^2+1)}$.
- (d) Evaluate $\int_C \overline{z} dz$ from z = 0 to z = 4 + 2ialong the curve C given by $z = t^2 + it$.
- 3. Answer **any three** questions from the following: 5×3=15
 - (a) Show that the three cube roots of -8i lie at the vertices of an equilateral triangle that is inscribed in a circle of radius 2 centred at the origin.
 - (b) Prove that $\lim_{z\to 0} \frac{\overline{z}}{z}$ does not exist.
 - (c) If f'(z) = 0 everywhere in a domain D, then prove that f(z) must be constant throughout D.
 - (d) Suppose that a function f(z) is analytic at a point $z_0 = z(t_0)$ on a differentiable arc $z = z(t)(a \le t \le b)$. Show that if w(t) = f(z(t)) then w'(t) = f'(z(t))z'(t) when $t = t_0$.

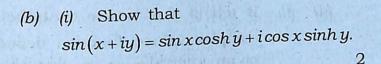
- (e) Using anti-derivative, evaluate the integral $\int_{c}^{z^{1/2}} dz$, where C is a contour from z=-3 to z=3 that, except for its end points, lies above the x-axis.
- 4. Answer **any three** questions from the following: 10×3=30
 - (a) (i) If z_0 and w_0 are points in the z and w planes respectively, then prove that
 - (A) $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$ and
 - (B) $\lim_{z\to\infty} f(z) = w_0$ if and only if

$$\lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$
 5

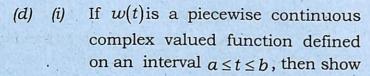
(ii) Let u and v denote the real and imaginary components of the function f defined by the equations:

$$f(z) = \begin{cases} \frac{(\overline{z})^2}{z}, & \text{when } z \neq 0 \\ 0, & \text{when } z = 0 \end{cases}$$

Verify the Cauchy-Riemann equations at the origin z = (0,0).



- (ii) Find numbers z = x + iy such that $e^z = 1 + i$.
- (iii) Show that if a function f(z) = u(x,y) + iv(x,y) and its conjugate $\overline{f(z)}$ are both analytic in a domain D, then f(z) must be constant throughout D.
- (c) (i) Show that the zeros of sin z are all real.
 - (ii) Evaluate $\int_{0}^{\pi/4} e^{it} dt$.
 - (iii) If $w = f(z) = \frac{1+z}{1-z}$ find $\frac{dw}{dz}$ and determine where f(z) is not analytic.



that
$$\left| \int_{a}^{b} w(t) dt \right| \leq \int_{a}^{b} |w(t)| dt$$

(ii) Let C denote the line segment from z = i to z = 1. By observing that of all the points on that line segment, the midpoint is the closest to the

origin, show that $\left| \int_{c} \frac{dz}{z^4} \right| \le 4\sqrt{2}$, without evaluating the integral.

(e) (i) Show that

$$\int \frac{dz}{z^2 + a^2} = \frac{1}{a} tan^{-1} \frac{z}{a} + C_1 = \frac{1}{2ai} ln \left(\frac{z - ai}{z + ai} \right) + C_2$$

(ii) Evaluate:

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz, \qquad 5$$

where C is the circle |z| = 3.

(ii) Let C denote the positively oriented boundary of a square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate

$$\int_{C} \frac{\tan(z/2)}{(z-x_0)^2} dz \ (-2 < x_0 < 2).$$
 5



⁽f) (i) Prove that if a function f is analytic at a given point, then its derivatives of all orders are also analytic at that point.