Total number of printed pages–7

3 (Sem–2/CBCS) MAT HC 1

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–2016

(Real Analysis)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : $1\times10=10$

   (a) Define, ε-neighbourhood of a point a in $\mathbb{R}$, where $\varepsilon > 0$.

   (b) Give an example of a set which is not bounded above.

   (c) Write the Archimedean property of $\mathbb{R}$.

(d) What is the limit of the sequence $\{x_n\}$, where $x_n = \dfrac{2n}{n^2+1}$?

(e) "The unit interval [0, 1] is uncountable."

(State True **or** False)

(f) For what value of p, the p-series $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges?

(g) "A convergent sequence of real numbers is a Cauchy sequence."

(State True **or** False)

(h) The sequence $\{0, 2, 0, 2, 0, 2, \ldots\}$ converges to the number 0.

(State True **or** False)

(i) Give an example of a Cauchy sequence in $\mathbb{R}$.

(j) Let $\{x_n\}$ be a non-zero sequence of real numbers such that $r = lim\left|\dfrac{x_{n+1}}{x_n}\right|$. Then Σx_n is absolutely convergent if

(i) $r < 1$

(ii) $0 \le r \le 1$

(iii) $r > 1$

(iv) $1 \le r \le 2$

(Choose the correct option)

2. Answer the following questions : 2×5=10

(a) Determine the set A of $x \in \mathbb{R}$ such that $|2x+3| < 7$.

(b) Define supremum of a non-empty subset S of $\mathbb{R}$. Write the supremum of the set $S = \{x \in \mathbb{R} : 0 < x < 1\}$.

(c) Show that the sequence $\left\{\dfrac{2n-7}{3n+2}\right\}$ is bounded.

(d) Show that the series $1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \ldots$ is convergent.

(e) State Cauchy's root test for convergence.

3. Answer ***any four*** questions : 5×4=20

(a) If $x > -1$, then prove that $(1+x)^n \geq 1+nx$ $\forall n \in \mathbb{N}$.

(b) If x and y are any real numbers with $x < y$, then show that there exists a rational number $r \in Q$ such that $x < r < y$.

(c) Prove that a convergent sequence of real numbers is bounded.

(d) Show that $\lim\limits_{n\to\infty}\left(n^{1/n}\right)=1$

(e) Use comparison test to test convergence of the series whose n^{th} term is $\frac{1}{\sqrt{n+1}}$.

(f) Show that every absolutely convergent series is convergent. Is the converse true ? Justify. 4+1=5

4. Answer the following questions : 10×4=40

(a) For $a, b \in \mathbb{R}$, prove that—

(i) $|a+b| \leq |a|+|b|$

(ii) $\big||a|-|b|\big| \leq |a-b|$

(iii) $|a-b| \leq |a|+|b|$ 5+3+2=10

Or

Prove that there exists a positive real number x such that $x^2 = 2$.

(b) (i) Prove that every monotonically increasing sequence which is bounded above converges to its least upper bound. 5

(ii) Show that a sequence in $\mathbb{R}$ can have atmost one limit. 5

Or

(i) Show that a bounded sequence of real numbers has a convergent subsequence. 5

(ii) State and prove that nested interval theorem. 5

(c) Suppose that $X = (x_n), Y = (y_n)$ and $Z = (z_n)$ are sequences of real numbers such that $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$ and that $\lim x_n = \lim z_n$. Then prove that $Y = (y_n)$ is convergent and $\lim(x_n) = \lim(y_n) = \lim(z_n)$. Use the result to show that $\lim_{n\to\infty}\left(\frac{\sin n}{n}\right) = 0$. 7+3=10

Or

Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}, n \in \mathbb{N}$ is convergent and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ where $2 < e < 3$.

(d) (i) Define absolute convergene of a series in $\mathbb{R}$. 2

(ii) Let Σu_n be any absolutely convergent series in $\mathbb{R}$. Then show that any rearrangement Σv_n of Σu_n converges to the same value. 7

(iii) Write one rearrangement of the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$. 1

Or

Discuss convergence of the geometric series $\sum_{n=0}^{\infty} r^n$, where $r \in \mathbb{R}$.
