

Vector Identities and related physical examples

September 19, 2026

Prove: $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}.$

Proof

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A}$$

$$= \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \vec{\nabla} \times \left[\hat{i} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{j} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{k} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \end{vmatrix}$$

$$\begin{aligned}
&= \hat{i} \left[\frac{\partial}{\partial y} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right] \\
&+ \hat{j} \left[\frac{\partial}{\partial z} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \frac{\partial}{\partial x} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] \\
&+ \hat{k} \left[\frac{\partial}{\partial x} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \right] \\
&= \hat{i} \left[\frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x} \right] \\
&+ \hat{j} \left[\frac{\partial^2 A_z}{\partial z \partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x \partial y} \right] \\
&+ \hat{k} \left[\frac{\partial^2 A_x}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial z} \right] \\
&= \hat{i} \left[\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y \partial x} + \frac{\partial^2 A_z}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial x^2} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} \right] \\
&+ \hat{j} \left[\frac{\partial^2 A_x}{\partial x \partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z \partial y} - \frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_y}{\partial y^2} - \frac{\partial^2 A_y}{\partial z^2} \right] \\
&+ \hat{k} \left[\frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial y \partial z} + \frac{\partial^2 A_z}{\partial z^2} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} - \frac{\partial^2 A_z}{\partial z^2} \right]
\end{aligned}$$

$$\begin{aligned}
&\nearrow \vec{\nabla} \cdot \vec{A} \qquad \qquad \qquad \nearrow \nabla^2 \\
&= \hat{i} \left[\frac{\partial}{\partial x} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) - \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) A_x \right] \\
&+ \hat{j} \left[\frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) - \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) A_y \right] \\
&+ \hat{k} \left[\frac{\partial}{\partial z} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) - \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \right) A_z \right] \\
&= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \vec{\nabla} \cdot \vec{A} - \nabla^2 \left(\hat{i} A_x + \hat{j} A_y + \hat{k} A_z \right)
\end{aligned}$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

Hence Proved.

Physical Problems:

1. Laplacian of Gravitational potential.

We know that the gravitational potential is given by

$$U_g = -\frac{Gm_1m_2}{r} \quad (1)$$

Where, G is the gravitational constant, m_1 is the mass of the first body, m_2 mass of the second body, and $\vec{r}$ is the distance between the centres of m_1 and m_2 .

Now,

$$\begin{aligned} \nabla^2 U_g &= \nabla^2 \left\{ -\frac{Gm_1m_2}{r} \right\} \\ &= -Gm_1m_2 \nabla^2 \left\{ \frac{1}{r} \right\} \end{aligned} \quad (2)$$

Now,

$$\begin{aligned} &\nabla^2 \left(\frac{1}{r} \right) \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) \end{aligned} \quad (3)$$

$$\begin{aligned}
\therefore \frac{\partial}{\partial x} \left\{ \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right\} &= -\frac{1}{2} \cdot 2x \{x^2 + y^2 + z^2\}^{-3/2} \\
\Rightarrow \frac{\partial^2}{\partial x^2} \left\{ \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right\} &= -x \left(-\frac{3}{2} \right) 2x \{x^2 + y^2 + z^2\}^{-5/2} \\
&\quad - \{x^2 + y^2 + z^2\}^{-3/2} \\
&= 3x^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2} \\
&= \frac{3x^2 - (x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \tag{4}
\end{aligned}$$

Similarly,

$$\frac{\partial^2}{\partial y^2} \left\{ \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right\} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \tag{5}$$

And,

$$\frac{\partial^2}{\partial z^2} \left\{ \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right\} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} \tag{6}$$

$$\begin{aligned}
&\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) \\
&= \frac{(2x^2 - y^2 - z^2) + (2y^2 - x^2 - z^2) + (2z^2 - x^2 - y^2)}{(x^2 + y^2 + z^2)^{5/2}} \\
&= 0
\end{aligned}$$

Hence,

$$\nabla^2 \left(\frac{1}{r} \right) = 0 \tag{7}$$

Therefore,

$$\nabla^2 U_g = 0. \tag{8}$$

2. Divergence of Central Force:

The general expression for central force field is,

$$\vec{F}(r) = f(r)\hat{r} = f(r)\frac{\vec{r}}{r}, \tag{9}$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector.

$$\begin{aligned}\therefore \vec{\nabla} \cdot \vec{F}(r) &= \vec{\nabla} \cdot \left(f(r) \frac{\vec{r}}{r} \right) \\ &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(f(r) \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} \right)\end{aligned}$$

Let, $\frac{f(r)}{r} = g(r)$

$$\vec{\nabla} \cdot \vec{F}(r) = \frac{\partial}{\partial x}(g(r)x) + \frac{\partial}{\partial y}(g(r)y) + \frac{\partial}{\partial z}(g(r)z)$$

Now,

$$\frac{\partial}{\partial x}(g(r)x) = g(r) + xg'(r)$$

Again,

$$\begin{aligned}g'(r) &= \frac{\partial g(r)}{\partial x} = \frac{\partial g(r)}{\partial r} \frac{\partial r}{\partial x} \\ &= \frac{\partial}{\partial r} \left\{ \frac{f(r)}{r} \right\} \frac{\partial r}{\partial x} \\ &= \left(\frac{f'(r)}{r} - \frac{f(r)}{r^2} \right) \frac{\partial r}{\partial x} \\ \therefore \frac{\partial}{\partial x}(g(r)x) &= \frac{f(r)}{r} + x \left[\frac{f'(r)}{r} - \frac{f(r)}{r^2} \right] \frac{\partial r}{\partial x} \\ &= \frac{f(r)}{r} + x \frac{f'(r)}{r} \frac{x}{r} - \frac{f(r)}{r^2} x \frac{x}{r} \\ &\quad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r} \right] \\ &= \frac{f(r)}{r} + f'(r) \frac{x^2}{r^2} - \frac{f(r)}{r} \frac{x^2}{r^2} \\ &= \frac{f(r)}{r} \left[1 - \frac{x^2}{r^2} \right] + f'(r) \frac{x^2}{r^2}\end{aligned}\tag{10}$$

Similarly,

$$\frac{\partial}{\partial y}(g(r)y) = \frac{f(r)}{r} \left[1 - \frac{y^2}{r^2} \right] + f'(r) \frac{y^2}{r^2}\tag{11}$$

And

$$\frac{\partial}{\partial z}(g(r)z) = \frac{f(r)}{r} \left[1 - \frac{z^2}{r^2} \right] + f'(r) \frac{z^2}{x^2} \quad (12)$$

$$\begin{aligned} \therefore \vec{\nabla} \cdot f(r)\hat{r} &= \frac{f(r)}{r} \left[1 - \frac{x^2}{r^2} + 1 - \frac{y^2}{r^2} + 1 - \frac{z^2}{r^2} \right] + \frac{f'(r)}{r^2} (x^2 + y^2 + z^2) \\ &= \frac{f(r)}{r} \left[3 - \frac{x^2 + y^2 + z^2}{r^2} \right] + \frac{f'(r)}{r^2} r^2 \\ &= 2 \frac{f(r)}{r} + f'(r) \end{aligned} \quad (13)$$

Thus, the divergence of central force is,

$$\vec{\nabla} \cdot f(r)\hat{r} = 2 \frac{f(r)}{r} + f'(r)$$

Homework:

Determine the divergence of gravitational force (or, inverse-square central force).

Ans: 0

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