VIRIAL THEOREM

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The gravitational potential energy of a stellar object could be transformed into its kinetic energy of motion. The random motion of gas particles provides the pressure that supports the star against self-gravitation. When the star contracts to a smaller radius, self-gravitation increases, so that the internal pressure also increases to maintain hydrostatic equilibrium. The ***Virial theorem*** provides the relation between gravitational potential energy and kinetic energy of a star. It states that

$$\boxed{2K + \Omega = 0}$$

Here, K is the kinetic energy, Ω is the gravitational potential energy.

Lets consider a configuration of N particles with mass m_i, position vector $\vec{r_i}$, velocity $\vec{v_i}$, and momentum $\vec{p_i}$. For this system the total moment of inertia is given by,

$$I = \sum_{i=1}^{N} m_i |\vec{r_i}|^2 = \sum_{i=1}^{N} m_i \vec{r_i} \cdot \vec{r_i} \tag{1}$$

Differentiating Eq. (1) w.r.t. time

$$\frac{dI}{dt} = 2 \sum_{i=1}^{N} m_i \vec{r_i} \cdot \frac{d\vec{r_i}}{dt}$$

$$\frac{1}{2}\frac{dI}{dt} = \sum_{i=1}^{N} m_i \vec{r_i} \cdot \dot{\vec{r}}_i \tag{2}$$

The time derivative of the moment of inertia is known as **virial**. Differentiating Eq. (2) again w.r.t. time we have

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \sum_{i=1}^{N} m_i \dot{\vec{r_i}}^2 + \sum_{i=1}^{N} m_i \vec{r_i} \cdot \ddot{\vec{r}}_i \tag{3}$$

$$= \sum_{i=1}^{N} m_i \vec{v_i}^2 + \sum_{i=1}^{N} m_i \vec{r_i} \cdot \vec{a_i} \tag{4}$$

Now from Newton's Second law, $\sum_{i=1}^{N} m_i \vec{a_i} = \vec{F_i}$.
Again, $\frac{1}{2}\sum_{i=1}^{N} m_i \vec{v_i}^2 = KE = K$ (say).

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \sum_{i=1}^{N} \vec{r_i} \cdot \vec{F_i} \tag{5}$$

Please turn over ...

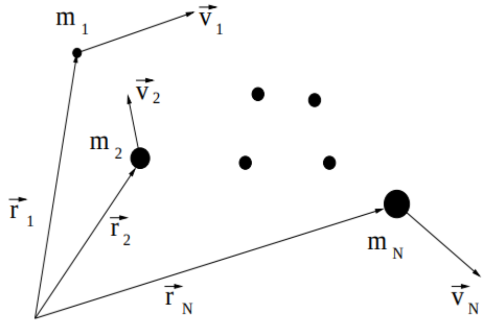

Figure 1: System of N particles.

If the N particles experience mutual gravitational force ($\vec{F_{gi}}$) and some external force ($\vec{F_{ef}}$) then the force on the i_{th} particle is

$$\vec{F_i} = \vec{F_{gi}} + \vec{F_{ef}}$$

From Eq.(5),

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2K + \sum_{i=1}^{N} \vec{r_i} \cdot \vec{F_{gi}} + \sum_{i=1}^{N} \vec{r_i} \cdot \vec{F_{ef}}. \tag{6}$$

Now, $\vec{F_{gi}} = \sum_j Gm_im_j \frac{\vec{r_j}-\vec{r_i}}{\vec{r_{ij}}^3}$

$$\therefore \sum_{i=1}^{N} \vec{r_i} \cdot \vec{F_{gi}} = \sum_{i=1}^{N}\sum_{j=1}^{N} Gm_im_j \frac{\vec{r_i} \cdot (\vec{r_j} - \vec{r_i})}{r_{ij}^3} \tag{7}$$

Interchanging i and j in the summation,

$$\therefore \sum_{j=1}^{N} \vec{r_j} \cdot \vec{F_{gj}} = \sum_{j=1}^{N}\sum_{i=1}^{N} Gm_jm_i \frac{\vec{r_j} \cdot (\vec{r_i} - \vec{r_j})}{r_{ij}^3}$$

$$= -\sum_{j=1}^{N}\sum_{i=1}^{N} Gm_im_j \frac{\vec{r_j} \cdot (\vec{r_j} - \vec{r_i})}{r_{ij}^3}$$

Adding the two equations, we get

$$\therefore \sum_{i=1}^{N} \vec{r_i} \cdot \vec{F_{gi}} = \sum_{j=1}^{N}\sum_{i=1}^{N} Gm_jm_i \frac{(\vec{r_i} - \vec{r_j}) \cdot (\vec{r_j} - \vec{r_i})}{r_{ij}^3} \tag{8}$$

$$= -\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} Gm_im_j \frac{(\vec{r_j} - \vec{r_i})^2}{r_{ij}^3} \tag{9}$$

$$= -\sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \frac{Gm_im_j}{r_{ij}} = \Omega. \tag{10}$$

where Ω is the gravitational potential energy. If the external forces act in the form of pressure at the boundary,

$$\begin{aligned}\sum_{i=1} \vec{r}_i \cdot \vec{F}_{ef} &= -\iint P d\vec{S} \cdot \vec{r} \\ &= -\iiint P \vec{\nabla} \cdot \vec{r} dV.\end{aligned}$$

Here $d\vec{S}$ and dV are surface and volume integration respectively. As $\vec{\nabla} \cdot \vec{r} = 3$

$$\therefore \sum_i \vec{r}_i \vec{F}_{ef} = -\iiint 3P dV = -3PV. \tag{11}$$

$\therefore$ from Eq.(6),(10), (11),

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2K + \Omega - 3PV. \tag{12}$$

For a system in equilibrium, dI/dt is minimum and $\frac{d^2 I}{dt^2} = 0$.

$$\boxed{\therefore 2K + \Omega = 3PV}. \tag{13}$$

In absence of external pressure,

$$\boxed{2K + \Omega = 0}. \tag{14}$$

This is the Virial Theorem.