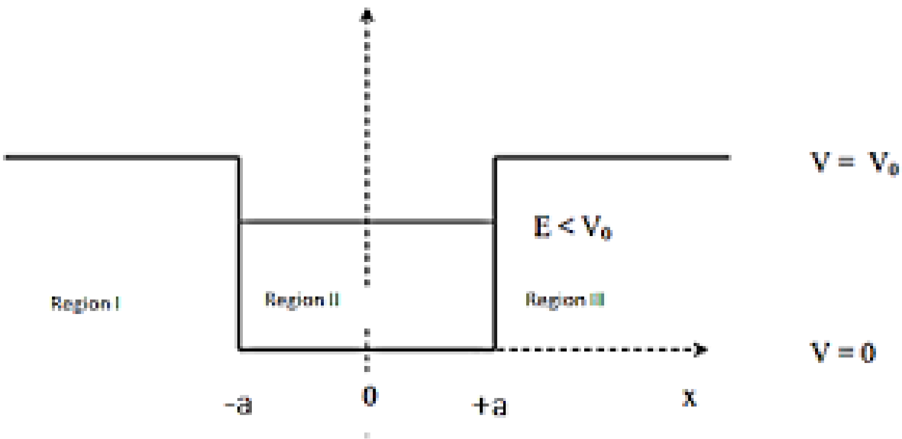
One Dimensional Potential Well of finite depth

March 2, 2026



Let's consider a particle of mass m and kinetic energy E be traveling parallel to the $x-axis$ and approaching a square-well potential of finite depth as shown in the figure. The potential will be represented by

$$\begin{aligned} V &= V_0 \quad for \quad x < -a \\ &= 0 \quad for \quad -a < x < a \\ &= V_0 \quad for \quad x > a \end{aligned} \tag{1}$$

Now, if ψ_1 is the wave function for the particle in Region I then the Schr$\ddot{o}$dinger's equation for the particle in Region I is,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} - (E - V_0)\psi_1 = 0; \qquad for \quad E < V_0$$

$$\frac{d^2\psi_1}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\psi_1 = 0$$

$$\frac{d^2\psi_1}{dx^2} - \beta^2\psi_1 = 0 \tag{2}$$

Here, $\beta^2 = 2m(V_0 - E)/\hbar^2$
If ψ_2 is the wave function for the particle in Region II then the time independent Schrödinger's equation for the particle is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} - E\psi_2 = 0$$

$$\frac{d^2\psi_2}{dx^2} + \frac{2m}{\hbar^2}E\psi_2 = 0$$

$$\frac{d^2\psi_2}{dx^2} + \alpha^2\psi_2 = 0 \tag{3}$$

Here, $\alpha^2 = 2mE/\hbar^2$.
Similarly, if ψ_3 is the wave function for the particle in Region III, then

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_3}{dx^2} - (E - V_0)\psi_3 = 0$$

$$\frac{d^2\psi_3}{dx^2} - \frac{2m}{\hbar^2}(V_0 - E)\psi_3 = 0$$

$$\frac{d^2\psi_3}{dx^2} - \beta^2\psi_3 = 0 \tag{4}$$

The general solution for the equations (2), (3), and (4) are

$$\begin{aligned} \psi_1 &= Ae^{\beta x} + Be^{-\beta x} \\ \psi_2 &= Ce^{i\alpha x} + De^{-i\alpha x} \\ \psi_3 &= Fe^{\beta x} + Ge^{-\beta x} \end{aligned} \tag{5}$$

Since the wave function must remain well-behaved, and therefore ψ should be zero at $x = \pm\infty$. Using this condition for RegioIII $F = 0$. Thus, we get

$$\begin{aligned} \psi_1 &= Ae^{\beta x} \\ \psi_2 &= Ce^{i\alpha x} + De^{-i\alpha x} \\ \psi_3 &= Ge^{-\beta x} \end{aligned} \tag{6}$$

The constants A, C, D, G can be calculated using following *Boundary conditions*,

1. ψ is continuous at $x = -a$ and $x = a$ i.e.

$$\psi_1|_{x=-a} = \psi_2|_{x=-a} \qquad\qquad \psi_2|_{x=a} = \psi_3|_{x=a}.$$

2. $\frac{d\psi}{dx}$ is continuous at $x = -a$, and $x = a$, i.e.

$$\frac{d\psi_1}{dx}|_{x=-a} = \frac{d\psi_2}{dx}|_{x=-a} \qquad\qquad \frac{d\psi_2}{dx}|_{x=a} = \frac{d\psi_3}{dx}|_{x=a}.$$

Applying the *Boundary conditions* in Equation (6),

$$Ae^{-\beta a} = Ce^{-i\alpha a} + De^{i\alpha a} \tag{7}$$

$$Ge^{-\beta a} = Ce^{i\alpha a} + De^{-i\alpha a} \tag{8}$$

$$\begin{aligned} \beta Ae^{-\beta a} &= i\alpha Ce^{-i\alpha a} - i\alpha De^{i\alpha a} \\ &= i\alpha(Ce^{-i\alpha a} - De^{i\alpha a}) \\ \frac{\beta}{i\alpha}Ae^{-\beta a} &= Ce^{-i\alpha a} - De^{i\alpha a} \end{aligned} \tag{9}$$

$$\begin{aligned} \beta Ge^{-\beta a} &= i\alpha Ce^{i\alpha a} - i\alpha De^{-i\alpha a} \\ &= i\alpha(Ce^{i\alpha a} - De^{-i\alpha a}) \\ -\frac{\beta}{i\alpha}Ge^{-\beta a} &= Ce^{i\alpha a} - De^{-i\alpha a} \end{aligned} \tag{10}$$

Adding Eq.(7) and Eq.(9) we get

$$\begin{aligned} 2Ce^{-i\alpha a} &= \left(1 + \frac{\beta}{i\alpha}\right) Ae^{-\beta a} \\ &= \left(\frac{\alpha - i\beta}{\alpha}\right) Ae^{-\beta a} \end{aligned} \tag{11}$$

Again, subtracting Eq.(9) from Eq.(7) we get

$$\begin{aligned} 2De^{i\alpha a} &= \left(1 - \frac{\beta}{i\alpha}\right) Ae^{-\beta a} \\ &= \left(\frac{\alpha + i\beta}{\alpha}\right) Ae^{-\beta a} \end{aligned} \tag{12}$$

Dividing Eq.(11) by Eq.(12)

$$\frac{C}{D} = \left(\frac{\alpha - i\beta}{\alpha + i\beta}\right) e^{2i\alpha a} \tag{13}$$

Similarly, adding Eq.(8) and Eq.(10) we get

$$\begin{aligned} 2Ce^{i\alpha a} &= \left(1 - \frac{\beta}{i\alpha}\right) Ge^{-\beta a} \\ &= \left(\frac{\alpha + i\beta}{\alpha}\right) Ge^{-\beta a} \end{aligned} \tag{14}$$

And, subtracting Eq.(10) from Eq.(8)

$$\begin{aligned} 2De^{-i\alpha a} &= \left(1 + \frac{\beta}{i\alpha}\right) Ge^{-\beta a} \\ &= \left(\frac{\alpha - i\beta}{\alpha}\right) Ge^{-\beta a} \end{aligned} \tag{15}$$

Dividing Eq.(14) by Eq.(15)

$$\frac{C}{D} = \left(\frac{\alpha + i\beta}{\alpha - i\beta}\right) e^{-2i\alpha a} \tag{16}$$

Multiplying Eq.(13) and Eq.(16)

$$\begin{aligned} \frac{C^2}{D^2} &= \left(\frac{\alpha - i\beta}{\alpha + i\beta}\right) e^{2i\alpha a} \times \left(\frac{\alpha + i\beta}{\alpha - i\beta}\right) e^{-2i\alpha a} \\ \Rightarrow \frac{C^2}{D^2} &= 1 \\ \Rightarrow \frac{C}{D} &= \pm 1 \end{aligned} \tag{17}$$

Now for $\frac{C}{D} = 1 \Rightarrow C = D$, we get From Eq.(7), and Eq.(8) $A = G$.
Again, since $C = D$, from Eq.(6)

$$\begin{aligned} \psi_2 &= C(e^{i\alpha x} + e^{-i\alpha x}) \\ &= 2C \cos \alpha x \end{aligned}$$

Thus, $\psi_2(-x) = \psi_2(x)$.
i.e. $\psi_2(x)$ has even parity i.e. $\psi_2(x)$ for E_0 and E_2 are symmetric about $x = 0$.
Also, for $\frac{C}{D} = -1 \quad \Rightarrow C = -D$, and thus $A = -G$ and,

$$\begin{aligned} \psi_2 &= C(e^{i\alpha x} - e^{-i\alpha x}) \\ &= 2C \sin \alpha x \end{aligned}$$

Thus, $\psi_2(-x) = -\psi_2(x)$.
i.e. $\psi_2(x)$ is antisymmetric about $x = 0$.

Energy eigenvalues: From Eq.(13) and Eq.(16);

$$\begin{aligned} \frac{\alpha - i\beta}{\alpha + i\beta} e^{2i\alpha a} &= \left(\frac{\alpha + i\beta}{\alpha - i\beta}\right) e^{-2i\alpha a} \\ \Rightarrow (\alpha + i\beta)^2 e^{-2i\alpha a} &= (\alpha - i\beta)^2 e^{2i\alpha a} \\ \Rightarrow (\alpha + i\beta) e^{-i\alpha a} &= \pm(\alpha - i\beta) e^{i\alpha a} \end{aligned} \tag{18}$$

The +ve sign on R.H.S. of Eq.(18) is related to the symmetric solution and the -ve sign to the anti-symmetric solution.
For a symmetric solution,

$$\begin{aligned} (\alpha + i\beta) e^{-i\alpha a} &= (\alpha - i\beta) e^{i\alpha a} \\ \Rightarrow \alpha[e^{i\alpha a} - e^{-i\alpha a}] &= i\beta[e^{-\alpha a} + e^{-i\alpha a}] \\ \Rightarrow \alpha(2i \sin \alpha a) &= i\beta(2 \cos \alpha a) \\ \Rightarrow \alpha \left(\frac{\sin \alpha a}{\cos \alpha a}\right) &= \beta \\ \Rightarrow \alpha a \tan(\alpha a) &= \beta a \end{aligned} \tag{19}$$

For an anti-symmetric solution;

$$\begin{aligned} (\alpha + i\beta) e^{-i\alpha a} &= -(\alpha - i\beta) e^{i\alpha a} \\ \Rightarrow \alpha[e^{i\alpha a} + e^{-i\alpha a}] &= i\beta[e^{-\alpha a} - e^{-i\alpha a}] \\ \Rightarrow \alpha(2 \cos \alpha a) &= i\beta(2i \sin \alpha a) \\ \Rightarrow \alpha \left(\frac{\cos \alpha a}{\sin \alpha a}\right) &= -\beta \\ \Rightarrow \alpha a \cot(\alpha a) &= -\beta a \end{aligned} \tag{20}$$

Again,

$$\begin{aligned} \alpha^2 + \beta^2 &= \frac{2mE}{\hbar^2} + \frac{2m(V_0 - E)}{\hbar^2} \\ &= \frac{2mV_0}{\hbar^2} \\ \Rightarrow (\alpha^2 + \beta^2) a^2 &= \frac{2mV_0 a^2}{\hbar^2} \end{aligned} \tag{21}$$

Substituting $p = \alpha a$, $q = \beta a$ in Eq.(19), Eq.(20), and Eq.(21)

$$p \tan p = q \tag{22}$$

$$p \cot p = -q \tag{23}$$

$$p^2 + q^2 = \frac{2mV_0 a^2}{\hbar^2} \tag{24}$$

Energy eigen values of the particle can be obtained by solving Eq.(22) and Eq.(24) for the symmetric case, and by solving Eq.(23), and Eq.(24) for anti-symmetric cases respectively. Since no analytical solution is possible, the only possible way to solve is the graphical solution. Since, α, and β are restricted to positive values, the required energy values will be given by the intersections of $q = p \tan p$ and $q = -p \cot p$ in the 1^{st} quadrant with the circle of known radius $(\frac{2mEV_0a^2}{\hbar^2})^{1/2}$.

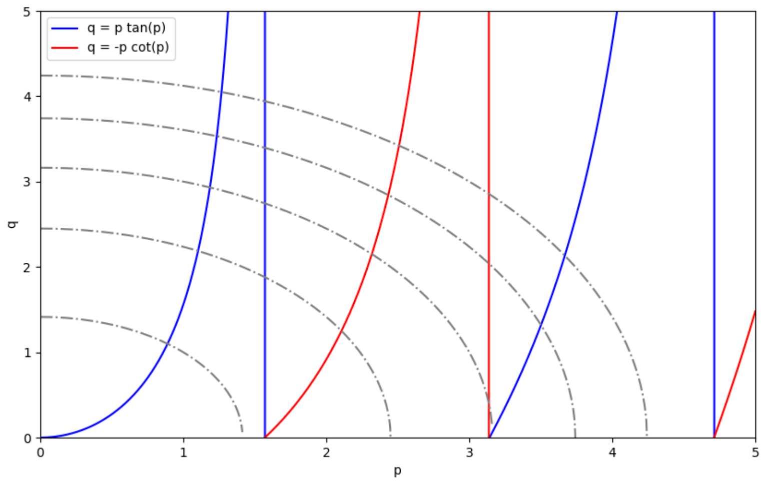


Fig. Graphical solution