Measurement of astronomical quantities

Luminosity: The total amount of energy (radiation) emitted by a star per unit second is called the luminosity of the star.

Let E be the surface energy radiated by a star per unit sec per unit surface area, then the luminosity is

$$L = 4\pi R^2 E \quad \text{........(1)}$$

Now, from Stefan's law, the energy radiated by the star per unit sec, per unit surface area is

$$E = \sigma T^4 \quad \text{.........(2)}$$

where, T is the surface temperature of the star.
Thus, the luminosity of the star is,

$$L = 4\pi R^2 \sigma T^4 \quad \text{..........(3)}$$

Brightness: The brightness of a star is defined with respect to an observer at a distance 'd' as the amount of energy passing normally through a unit area in unit second at the location of the observer.
Let L be the luminosity of a star and b be its brightness for an observer at a distance d from the Sun. Then,

$$b = \frac{L}{4\pi d^2} \quad \text{..........(4)}$$

Luminosity is related to the star only, whereas brightness is related to the observer. If the distance of the star is increased, its brightness diminishes by the square of its distance.

Stellar magnitude scale:
Greek astronomers Hipparchus and Ptolemy divided stars into classes according to their brightness, and their basic subdivisions still stand today. They categorized the brightest stars of a night sky as first-magnitude stars. Slightly fainter than the 1st magnitude stars are categorized as the 2nd magnitude stars, and so on. The faintest stars that can be seen by the naked eye are categorized as the 6th magnitude stars. This is known as the ***stellar magnitude scale***.

Later, it was realized that the brightest star is 100 times brighter than the faintest star that could be seen by the naked eye. This corresponds to a difference of 5 magnitude on

Ptolemy-Hipperchus scheme. Thus, a star of 1st magnitude is exactly 100 times brighter than a star of 6th magnitude.

Brightness ratio and magnitude:

Let, b_1 is the brightness of the star with magnitude m_1, and b_2 is the brightness of the star with magnitude m_2.

Since the 1st magnitude stars are 100 times brighter than the 6th magnitude stars.

$$\therefore \frac{b_1}{b_2} = 100$$

$$\Rightarrow \frac{b_1}{b_2} = \left(10^{2/5}\right)^5 = \left[10^{2/5}\right]^{6-1} = \left[10^{2/5}\right]^{m_2-m_1}$$

$$\Rightarrow \frac{b_1}{b_2} = 10^{\left(m_2-m_1\right)/2.5}$$

Taking the logarithm of both sides,

$$log\left(\frac{b_1}{b_2}\right) = \frac{m_2-m_1}{2.5}$$

$$\Rightarrow m_2 - m_1 = 2.5\, log\left(\frac{b_1}{b_2}\right) \qquad ....(5)$$

This is the convention used to measure the stellar magnitude. Knowing the brightness ratio, we would be able to calculate the magnitude difference.

Absolute magnitude and distance modulus:

Let's consider two stars of brightness b_1 and b_2 are at distances of d_1 and d_2 from us. Suppose their apparent magnitudes are m_1, and m_2 respectively. Then, from the convention of stellar magnitude, we have,

$$m_2 - m_1 = 2.5\, log\left(\frac{b_1}{b_2}\right) \qquad ....(6)$$

Now, for the same star observed from two different distances d_1, and d_2 Eq (6) will be the same. In this situation, $b_1$and m_1 are the magnitude and brightness of the star observed from a distance of d_1, whereas, b_2, and m_2 are the magnitude and brightness of the same star observed from a distance of d_2 , respectively. If L is the luminosity of the star, then

$$b_1 = \frac{L}{4\pi d_1^2}, \text{ and } \quad b_2 = \frac{L}{4\pi d_2^2} \qquad ....(7)$$

Therefore, from Eq (6),

$$m_2 - m_1 = 2.5\, log\left(\frac{L/4\pi d_1^2}{L/4\pi d_2^2}\right)$$

$$= 2.5 log\left(\frac{d_2^2}{d_1^2}\right)$$

$$\therefore m_2 - m_1 = 5\, log\left(\frac{d_2}{d_1}\right) \qquad \text{..........(8)}$$

The magnitude measured at a distance of 10 pc is known as the absolute magnitude (M) of a star. Therefore, Eq (8) can be rewritten as

$$m - M = 5\, log\left(\frac{d}{10\, pc}\right) \qquad \text{.......... (9)}$$

where m is the apparent magnitude of the star at a distance d.

Thus, from Eq (9)

$$log\left(\frac{d}{10\, pc}\right) = \frac{m-M}{5}$$

$$\Rightarrow \frac{d}{10\, pc} = 10^{(m-M)/5}$$

$$\therefore d = 10\, pc \times 10^{(m-M)/5} \qquad \text{.........(10)}$$

If the apparent and absolute magnitudes of a star are known, one can calculate the distance of the star using Eq (10). Therefore, this is an important equation for distance measurement.

The difference between the apparent magnitude and absolute magnitude is known as the Distance modulus.

The magnitude of a star at a distance of 10 pc is defined as the absolute magnitude of the star.

Sample question:

1. Define the absolute magnitude of a star. Obtain the relation connecting the absolute magnitude and the distance of a star in parsec.
2. Starting with the basic concept of magnitude, derive the equation for the magnitude scale. What is a distance modulus?
3. Star A has a magnitude of +5, and star B has a magnitude of +10. Which one is brighter and how much?
4. The apparent magnitude of the full Moon is -12.5, and that of Venus is -4. Which one is brighter and how much?
5. Define Luminosity and radiant flux. Using Stefan-Boltzmann law of radiation, obtain the ratio of radii R_1 and R_2 of two stars with surface temperatures T_1 and T_2 and absolute magnitudes M_1 and M_2, respectively.

Temperature & Radius of a celestial object:

The laws of black-body radiation can be applied to the continuous spectrum of a star. Suppose we can observe the radiation in all wavelengths in absolute units. In that case, the Stefan-Boltzmann law of black body radiation can be used to obtain the effective temperature of the star. *The effective temperature of a star is defined as the temperature of a black body that would emit the same total amount of electromagnetic radiation.*

Suppose, L is the luminosity of a star, then one can obtain the relation between T_e, L, and R from

$$L = 4\pi R^2 \sigma T_{eff}^4 \qquad (1)$$

Here R is the radius of the star, T_{eff} is the effective temperature of the star, and σ is the Stefan's constant.

If the luminosity of a star is unknown, then one can use another tool to calculate the effective temperature (T_{eff}). This is the bolometric magnitude. The magnitude corresponding to the total luminosity of a star in all wavelengths is called bolometric magnitude (M_{bol}). The relation between the bolometric magnitude and luminosity is,

$$M_{bol} - M_{bol,\odot} = -\ 2.5\ log(L/L_{\odot}) \qquad (2)$$

where, $M_{bol,\odot}$, and $L_{\odot}$ are the bolometric magnitude and luminosity of the Sun, respectively.

$$\therefore\ M_{bol} - M_{bol,\odot} = -\ 2.5\ log\left(\frac{R_*^2 T_{eff,*}^4}{R_{\odot}^2 T_{eff,\odot}^4}\right) \qquad (3)$$

where, R_* and $T_{eff,*}$ are the radius and effective temperature of the star, respectively.

Thus, this equation can be used to calculate T_{eff} if R is known and vice versa.

Stellar Mass:

The mass of a body can be obtained from its gravitational attraction to other bodies. To obtain the mass of any celestial object, it must have a companion whose motion is governed by their mutual gravitational effect. For example, the mass of the Sun can be calculated by studying the motion of a planet around it. Similarly, the mass of a planet can be derived from the motion of its satellite. In the case of stars, we have to study the binary systems in which the two stars move around

their common center of mass. In all these cases, to calculate the mass, Kepler's law and Newton's theory of gravitation are used.
Let's consider two bodies in circular orbits about each other, with masses m_1, and m_2 and separated by a distance a. Suppose the center of mass lies at O, and the distance between the center of mass and the body of mass m_1 is a_1, and that between the center of mass and the body of mass m_2 is a_2.
Now, the center of mass is defined by,

$$m_1 a_1 = m_2 a_2$$

$$\Rightarrow a_1 = \frac{m_2 a_2}{m_1} \qquad (1)$$

Again, the total separation between the two bodies is

$$a = a_1 + a_2 = \frac{m_2 a_2}{m_1} + a_2$$

$$\Rightarrow a = \left(\frac{m_2 + m_1}{m_1}\right) a_2$$

$$\Rightarrow a_2 = \left(\frac{m_1}{m_2 + m_1}\right) a \qquad (2)$$

Now, the forces acting on the body of mass m_2 are gravitational attraction and centripetal attraction. By equating these two forces of attraction, we get,

$$F_g = F_{cp}$$

$$\Rightarrow \frac{G m_1 m_2}{a^2} = m_2 a_2 \omega^2,$$

where, G and ω are the gravitational constant and the angular frequency, respectively.

$$\Rightarrow \frac{G m_1 m_2}{a^2} = m_2 \left(\frac{m_1}{m_1 + m_2}\right) a\omega^2 = \left(\frac{m_1 m_2}{m_1 + m_2}\right) a\omega^2$$

$$\Rightarrow \frac{G}{a^3} = \left(\frac{1}{m_1 + m_2}\right) \omega^2$$

If P is the period required by the body of mass m_2 to orbit the massive body, then,

$$P = \frac{2\pi}{\omega} \qquad \Rightarrow \omega = \frac{2\pi}{P}$$

$$\therefore \frac{G}{a^3} = \left(\frac{1}{m_1 + m_2}\right) \left(\frac{2\pi}{P}\right)^2$$

$$\Rightarrow \frac{a^3}{P^2} = \frac{G(m_1 + m_2)}{4\pi^2} \qquad (3)$$

If the sum of the masses of the two stars is equal to that of the Earth and the Sun, $P = 1$ year, then the separation between the stars is 1AU. Thus, if we express P in the year, a in AU, and mass as the multiple of the solar mass, then we get

$$\frac{a^3}{P^2} = m_1 + m_2 \qquad (4)$$

Thus, using equation (3), or (4), the total mass of the binary system can be calculated. Additionally, using relation (1), the individual mass of the stars can be calculated.

If v_1, and v_2 are the orbital velocities of the stars then,

$$v_1 = \frac{2\pi a_1}{P} \Rightarrow a_1 = \frac{Pv_1}{2\pi},$$

and $v_2 = \frac{2\pi a_2}{P} \Rightarrow a_2 = \frac{Pv_2}{2\pi}$

Thus, $\frac{a_1}{a_2} = \frac{v_1}{v_2}$ (5)

Again from equation (1)

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}$$

$$\Rightarrow \frac{m_2}{m_1} = \frac{v_1}{v_2} \qquad (6)$$

Thus, if the orbital velocities of the individual stars are known, the individual masses can be calculated using equation (6).

Sample Questions:

1. The separation between two stars in some binary system is 0.5 AU. and the period is 1.1 year. What is the sum of the masses of the two stars?
2. For the binary star system of problem 1, if one of the masses has an orbital velocity twice that of the other, then what are their individual masses?
3. Show that if you double the distance between two bodies orbiting each other, the period increases by a factor of 2.8.

Mass-Luminosity Relation:

The mass–luminosity relation is an empirical and theoretical relationship between a star's mass and its luminosity. For the main-sequence stars, the relation is

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^{3.5}$$

Mass-Radius Relation:

The mass–radius relation is an empirical and theoretical relationship between a star's mass and its radius. For the main-sequence stars, the relation is

$$\frac{R}{R_{\odot}} = \left(\frac{M}{M_{\odot}}\right)^{0.75}$$

Mass-Temperature Relation:

The mass–temperature relation is an empirical and theoretical relationship between a star's mass and its temperature. For the main-sequence stars, the relation is

$$\frac{T_e}{T_{e,\odot}} = \left(\frac{M}{M_{\odot}}\right)^{0.5}$$

Luminosity-Temperature Relation:

The luminosity-temperature relation is an empirical and theoretical relationship between a star's luminosity and its temperature. For the main-sequence stars, the relation is

$$\frac{L}{L_{\odot}} = \left(\frac{T_e}{T_{e,\odot}}\right)^{6.9}$$

Sample question:

1. The mass of Sirius B is thrice that of the Sun. Find the ratio of luminosity and the difference in their absolute magnitude. Taking the absolute magnitude of the Sun as 5, find the absolute magnitude of Sirius B.
