Total number of printed pages–8

3 (Sem–5/CBCS) MAT HC 2

2023

MATHEMATICS

(Honours Core)

Paper : MAT–HC–5026

(Linear Algebra)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions as directed : 1×10=10

(a) Let $A = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}$ and $\vec{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$.

Check whether $\vec{u}$ is in null space of A.

(b) Define subspace of a vector space.

(c) Give reason why $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$.

Contd.

(d) State whether the following statement is true **or** false :

"If dimension of a vector space V is $p > 0$ and S is a linearly dependent subset of V, then S contains more than p elements."

(e) If $\bar{x}$ is an eigenvector of A corresponding to the eigenvalue λ then what is $A^3\bar{x}$?

(f) When two square matrices A and B are said to be similar?

(g) If $\bar{v} = (1\ -2\ 2\ 4)$ then find $\|\bar{v}\|$.

(h) Find a unit vector in the direction of $\bar{u} = \begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$.

(i) Under what condition two vectors $\bar{u}$ and $\bar{v}$ are orthogonal to each other?

(j) Define orthogonal complement of vectors.

2. Answer the following questions : 2×5=10

(a) Show that the set $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}$ is not a subspace of $\mathbb{R}^2$.

(b) Let $\vec{b_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{b_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\beta = \{b_1, b_2\}$. Find the coordinate vector $[x]_\beta$ of $\bar{x}$ relative to β.

(c) Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

(d) Let P_2 be the vector space of all polynomials of degree less than equal to 2. Consider the linear transformation $T : P_2 \rightarrow P_2$ defined by $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$. Find the matrix representation $[T]_\beta$ of T with respect to the base $\beta = \{1, t, t^2\}$.

(e) Show that the matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$ has orthogonal columns.

3. Answer ***any four*** questions : 5×4=20

(a) Let $S = \{v_1, v_2, \ldots, v_p\}$ be a set in the vector space V and $H = span(S)$. Now if one of the vector in S, say v_k, is linear combination of the other vectors in S, then show that S is linearly dependent and the subset of $S_1 = S - \{v_k\}$ still span H. 2+3=5

(b) Show that the set of all eigenvectors corresponding to the distinct eigenvalues of a $n \times n$ matrix A is linearly independent.

(c) Let W be a subspace of the vector space V and S is a linearly independent subset of W. Show that S can be extended, if necessary, to form a basis for W and $\dim W \leq \dim V$.

(d) If $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(e) If $\bar{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\bar{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$ then find the orthogonal projection of $\bar{y}$ onto $\bar{u}$ and write $\bar{y}$ as the sum of two orthogonal vectors, one in $span\{\bar{u}\}$ and the other orthogonal to $\bar{u}$.

(f) If $W = span\{x_1, x_2\}$ where $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$, find a orthogonal basis for W.

Answer ***either*** *(a)* ***or*** *(b)* from each of the following questions : 10×4=40

4. *(a)* Find a spanning set for the null space of the matrix :

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Is this spanning set linearly independent? 8+2=10

(b) (i) If a vector space V has a basis of n vectors, then show that every basis of V must consist of exactly n vectors. 4

(ii) Find a basis for column space of the following matrix : 6

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

5. (a) Define eigenvalue and eigenvector of a matrix. Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$. 2+8=10

(b) Let T be a linear operator on a finite dimensional vector space V and let W denote the *T-cyclic* subspace of V generated by a non-zero vector $v \in V$. If $dim(W) = k$ then show that

(i) $\{v, T(v), T^2(v), \ldots, T^{k-1}(v)\}$ is a basis for W.

(ii) If

$$a_0 v + a_1 T(v) + \ldots + a_{k-1} T^{k-1}(v) + T^k(v) = 0,$$

then the characteristics polynomial of T_w is

$$f(t) = (-1)^k \left(a_0 + a_1 t + \ldots + a_{k-1} t^{k-1} + t^k\right).$$

6+4=10

6. (a) (i) Define orthogonal set of vectors. Let $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, then show that S is linearly independent. 1+4=5

(ii) For any symmetric matrix show that any two eigenvectors from different eigenspaces are orthogonal. 5

(b) Define inner product space. Show that the following is an inner product in $\mathbb{R}^2$

$$\langle u, v \rangle = 4u_1v_1 + 5u_2v_2$$

Where $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$

Also, show that in any inner product space V,

$$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|, \ \forall u, v \in V.$$

2+4+4=10

7. *(a)* *(i)* Consider the bases $\beta = \{b_1, b_2\}$ and $\gamma = \{c_1, c_2\}$ for $\mathbb{R}^2$ where

$$b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$$

and $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, find the change of coordinate matrix from γ to β and from β to γ. 5

(ii) Compute A^{10} where

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}.$$ 5

(b) State Cayley-Hamilton theorem for matrices. Verify the theorem for the matrix $M = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and hence find M^{-1}.