Total number of printed pages–7

3 (Sem–3 /CBCS) MAT HC 1

2021

(Held in 2022)

MATHEMATICS

(Honours)

Paper : MAT-HC-3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×10=10

(a) Find $\lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1}$

(b) Is the function $f(x) = x \sin\left(\frac{1}{x}\right)$ continuous at $x=0$?

(c) Write the cluster points of $A = (0,1)$.

Contd.

(d) If a function $f: (a, \infty) \to \mathbb{R}$ is such that $\lim_{x\to\infty} xf(x) = L$, where $L \in \mathbb{R}$, then $\lim_{x\to\infty} f(x) = ?$

(e) Write the points of continuity of the function $f(x) = \cos\sqrt{1+x^2}$, $x \in \mathbb{R}$.

(f) "Every polynomial of odd degree with real coefficients has at least one real roof." Is this statement true ***or*** false ?

(g) The derivative of an even function is ________ function. *(Fill in the blank)*

(h) Between *any two* roots of the function $f(x) = \sin x$, there is at least ________ root of the function $f(x) = \cos x$.

(Fill in the blank)

(i) If $f(x) = |x^3|$ for $x \in \mathbb{R}$, then find $f'(x)$ for $x \in \mathbb{R}$.

(j) Write the number of solutions of the equation $\ln(x) = x - 2$.

2. Answer the following questions : 2×5=10

(a) Show that $\lim_{x\to 0}(x + sgn(x))$ does not exist.

(b) Let f be defined for all $x \in \mathbb{R}$, $x \neq 3$ by $f(x) = \dfrac{x^2 + x - 12}{x - 3}$. Can f be defined at $x = 3$ in such a way that f is continuous at this point ?

(c) Show that $f(x) = x^2$ is uniformly continuous on $[0, a]$, where $a > 0$.

(d) Give an example with justification that a function is 'continuous at every point but whose derivative does not exist everywhere'.

(e) Suppose $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin\dfrac{1}{x^2}$, for $x \neq 0$ and $f(0) = 0$. Is f' bounded on $[-1, 1]$?

3. Answer **any four** parts : 5×4=20

(a) If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then prove that f is bounded on some neighbourhood of c.

(b) Let $f(x) = |2x|^{-\frac{1}{2}}$ for $x \neq 0$. Show that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = +\infty.$$

(c) Show that the function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$.

(d) Give an example to show that the product of two uniformly continuous function is not uniformly continuous on $\mathbb{R}$.

(e) Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. If f' is positive on $[a, b]$, then prove that f is strictly increasing on $[a, b]$.

(f) Evaluate —

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$$

4. Answer **any four** parts : 10×4=40

(a) Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Prove that the following are equivalent.

(i) $\lim_{x \to c} f(x) = l$

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $x \in \mathbb{N}$, the sequence $(f(x_n))$ converges to l. 10

(b) (i) Give examples of functions f and g such that f and g do not have limits at a point c but such that both $f + g$ and fg have limits at c. 6

(ii) Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let c be a cluster point of A. If $\lim_{x \to c} f(x)$ exists and if $|f|$ denotes the function defined for $x \in A$ by $|f|(x) = |fx|$, Proof that

$$\lim_{x \to c} |f|(x) = \left| \lim_{x \to c} f(x) \right|$$ 4

Contd.

(c) Prove that the rational functions and the sine functions are continuous on $\mathbb{R}$. 10

(d) (i) Let I be an interval and let $f: I \to \mathbb{R}$ be continuous on I. Prove that the set $f(I)$ is an interval. 5

(ii) Show that the function $f(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. 5

(e) State and prove maximum-minimum theorem. 2+8=10

(f) (i) If $f: I \to \mathbb{R}$ has derivative at $c \in I$, then prove that f is continuous at c. Is the converse true ? Justify. 6

(ii) If r is a rational number, let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Determine those values of r for which $f'(0)$ exists. 4

(g) State and prove Mean value theorem. Give the geometrical interpretation of the theorem. (2+5)+3=10

(h) State and prove Taylor's theorem.

4500