2018

MATHEMATICS

(Major)

Paper : 3.2

(Linear Algebra and Vector)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following as directed : 1×7=7

(a) Describe geometrically the linear dependence of any two vectors u and v in the vector space R^3.

(b) Prove that if two vectors in a vector space V over the field F are linearly dependent, then one of them is a scalar multiple of the other.

(c) Let U and W be the following subspaces of R^3 :

$$U = \{(a,\ b,\ c) : a = b = c\} \text{ and } W = \{(0,\ b,\ c)\}$$

Clearly any $v = (a,\ b,\ c) \in U \cap W \Rightarrow a = 0$, $b = 0,\quad c = 0 \Rightarrow U \cap W = \{0\}$. Observing this, choose the correct option :

(i) $R^3 = U \oplus W$

(ii) $R^3 \neq U \oplus W$

(d) If U and V be two vector spaces over the same field F with $\dim U = m$ and $\dim V = n$, then the set $\text{Hom}(U,\ V)$ of all linear transformations from U to V is a vector space of dimension

(i) $m + n - 1$

(ii) $1 - (m + n)$

(iii) mn

(iv) $m + n$

(Choose the correct option)

(e) If T is a linear operator, then the following are equivalent :

(i) A scalar λ is an eigenvalue of T.

(ii) The linear operator $\lambda I - T$ is singular.

(Write true or false)

(f) Find the minimal polynomial $m(t)$ of the following matrix :

$$A = \begin{bmatrix} 5 & 1 \\ 3 & 7 \end{bmatrix}$$

(g) If λ is an eigenvalue of a linear operator (matrix) A, what is meant by the geometric multiplicity of λ?

2. Answer the following questions : 2×4=8

(a) Give an example of an infinite-dimensional vector space V with a subspace W such that the quotient space V/W is a finite-dimensional vector space.

(b) Suppose a linear transformation $T: V \to U$ is one-to-one and onto. Show that the inverse mapping $T^{-1}: U \to V$ is also a linear transformation.

(c) Consider the two bases of the vector space $R^2(R)$:

$$B_1 = \{(1, 2), (3, 5)\} \text{ and } B_2 = \{(1, -1), (1, -2)\}$$

Find the change-of-basis matrix M from B_1 to the 'new' basis B_2.

(d) If λ be an eigenvalue of a linear operator $T: V \to V$, then prove that the set E_λ of all eigenvectors belonging to λ is a subspace of V.

3. Answer any *one* part : 5

(a) Let V_1 and V_2 be vector spaces over the same field F and T be a linear transformation from V_1 into V_2. Show that if V_1 is finite dimensional, then $\text{rank}(T) + \text{nullity}(T) = \dim V_1$.

(b) Define kernel of a linear transformation. Find the range, rank, null space and nullity of the linear transformation $T: R^2 \to R^3$ such that $T(x_1, x_2) = (x_1, x_1 + x_2, x_2)$.

4. Answer the following questions : 10×2=20

(a) When is a subspace of a vector space V said to be spanned by a subset X of V? If U be a vector space which is spanned by a finite set of vectors $u_1, u_2, \ldots, u_m$ in U, then prove that any linearly independent set of vectors in U is finite and contains no more than m elements. 1+9=10

Or

If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then prove that $W_1 + W_2$ is also finite-dimensional and

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$ 10

(b) *(i)* Let V be a vector space over the field F and T be a linear operator on V. Define a characteristic value of T, a characteristic vector of T and the characteristic space associated with a characteristic value of T.

(ii) If T_1 and T_2 be linear operators on R^2 and C^2 respectively which are represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then find characteristic polynomials and characteristic values (if possible) for T_i (or for A), $i = 1, 2$.

(iii) Prove that similar matrices have the same characteristic polynomial.

3+3+4=10

Or

State the Cayley-Hamilton theorem and define the minimal polynomial of a matrix (linear operator) A. Find the minimal polynomial of

$$A = \begin{bmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{bmatrix}$$

1+1+8=10

GROUP—B

(Vector)

(*Marks* : 40)

5. Answer the following : 1×3=3

(a) Prove that the value of a scalar triple product, if two of its vectors are parallel, is zero.

(b) Prove that $\vec{a}\cdot\vec{b}\times\vec{c}=\vec{a}\times\vec{b}\cdot\vec{c}$.

(c) If $\vec{a}$ and $\vec{b}$ lie in a plane normal to the plane containing $\vec{c}$ and $\vec{d}$, then show that

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=0$$

6. Find the volume of the parallelepiped whose edges are represented by

$$\vec{a}=2\vec{i}-3\vec{j}+4\vec{k},\quad \vec{b}=\vec{i}+2\vec{j}-\vec{k}$$
$$\vec{c}=3\vec{i}-\vec{j}+2\vec{k}$$ 2

7. Answer the following questions : 5×3=15

(a) If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ are reciprocal systems of vectors, then prove that

$$\vec{a}'\times\vec{b}'+\vec{b}'\times\vec{c}'+\vec{c}'\times\vec{a}'=\frac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$$ 5

(b) *(i)* If $\vec{r} = \vec{a}\sin\omega t + \vec{b}\cos\omega t + \frac{\vec{c}t}{\omega^2}\sin\omega t$,

then prove that

$$\frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = \frac{2\vec{c}}{\omega}\cos\omega t$$

(ii) Give the geometrical interpretation of

$$\vec{r} \times \frac{d\vec{r}}{dt} = \vec{0}$$ 4+1=5

(c) Prove that

$$\operatorname{div}(\vec{A} \times \vec{B}) = \vec{B} \cdot \operatorname{curl}\vec{A} - \vec{A} \cdot \operatorname{curl}\vec{B}$$ 5

Or

When is a vector $\vec{f}$ said to be irrotational? Find the constants a, b, c so that the vector

$$\vec{f} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$$

is irrotational. 1+4=5

8. Answer the following questions : 10×2=20

(a) *(i)* If

$$\vec{a} = \sin\theta\vec{i} + \cos\theta\vec{j} + \theta\vec{k}$$
$$\vec{b} = \cos\theta\vec{i} - \sin\theta\vec{j} - 3\vec{k}$$
$$\vec{c} = 2\vec{i} + 3\vec{j} - 3\vec{k}$$

find $\frac{d}{d\theta}\{\vec{a} \times (\vec{b} \times \vec{c})\}$ at $\theta = \frac{\pi}{2}$.

(ii) Show that if $\vec{a}$, $\vec{b}$, $\vec{c}$ are constant vectors, then $\vec{r} = \vec{a}t^2 + \vec{b}t + \vec{c}$ is the path of a particle moving with constant acceleration. 7+3=10

Or

(i) Prove that the necessary and sufficient condition for a vector $\vec{v}(t)$ to be constant is that $\frac{d\vec{v}}{dt} = \vec{0}$.

(ii) If $\vec{r} \times d\vec{r} = \vec{0}$, show that $\hat{r} = \text{constant}$. 7+3=10

(b) If

$$\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k},$$

evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the line curve consisting of the straight lines from (0, 0, 0) to (1, 0, 0), then to (1, 1, 0) and then to (1, 1, 1). 10

Or

Evaluate $\iiint_V \vec{F} dV$ where $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ and V is the region bounded by the surfaces $x = 0$, $x = 2$, $y = 0$, $y = 6$, $z = 4$ and $z = x^2$. 10

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