2014

MATHEMATICS

(Major)

Paper : 4.1

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×10=10

(a) What are the limit points of the sequence $\{S_n\}$ where $S_n = (-1)^n \left(1+\frac{1}{n}\right), n \in N$?

(b) If $\sum_{n=1}^{\infty} u_n$ is a positive term series such that $\lim_{n \to \infty} (u_n)^{\frac{1}{n}} = l,$ then the series converges, if

(i) $l > 1$

(ii) $l < 1$

(iii) $l = 1$

(iv) None of the above

(Choose the correct answer)

(c) The series $1+r+r^2+r^3+\ldots$ is oscillatory, if

(i) $r<1$ (ii) $r=1$

(iii) $r>1$ (iv) $r=-1$

(Choose the correct answer)

(d) Write whether the following statement is True or False :

A series obtained from an absolutely convergent series by a rearrangement of terms converges absolutely and has the same sum as the original series.

(e) For what value of p, the series

$$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots$$

is divergent?

(f) Fill in the blank :

If the series $\sum_{n=1}^{\infty} u_n$ is convergent and $\sum_{n=1}^{\infty} |u_n|$ is not convergent, then $\sum_{n=1}^{\infty} u_n$ is known as ——.

(g) Find $\lim_{x\to 0^-} \frac{x-|x|}{x}$ and $\lim_{x\to 0^+} \frac{x-|x|}{x}$.

(h) Let

$$f(x)=\begin{cases} x, & \text{when } 0\le x<\frac{1}{2} \\ 1, & \text{when } x=\frac{1}{2} \\ 1-x, & \text{when } \frac{1}{2}<x<1 \end{cases}$$

Then f is

(i) continuous at $x=\frac{1}{2}$

(ii) not defined at $x=\frac{1}{2}$

(iii) discontinuous at $x=\frac{1}{2}$

(iv) continuous for all x, $0\le x<1$

(Choose the correct answer)

(i) Give an example of a continuous function which is not uniformly continuous.

(j) Consider the following statements :

I. A differentiable function is continuous.

II. A continuous function is differentiable.

III. A continuous function on a closed interval is uniformly continuous on that interval.

IV. A continuous function on a closed interval is not bounded in that interval.

Which of these statements are correct?

(i) I and II *(ii)* I and III

(iii) II and III *(iv)* II and IV

(Choose the correct answer)

2. Answer the following questions : 2×5=10

(a) Define neighbourhood of a point in $\mathbb{R}$ and an open subset of $\mathbb{R}$.

(b) Illustrate with an example that the sequences $\{a_n + b_n\}$ and $\{a_n b_n\}$ are convergent but the component sequences $\{a_n\}$ and $\{b_n\}$ may not be convergent.

(c) Show that

$$\lim_{x \to 0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

does not exist.

(d) Show that

$$\frac{x}{1+x} < \log x$$

for all $x > 0$.

(e) Find the value of c of Lagrange's mean value theorem, when

$$f(x) = 2x^2 - 7x + 10 \text{ in } [2, 5]$$

3. Answer any *four* parts : 5×4=20

(a) Prove that the union of two closed sets is a closed set. Give an example to show that the union of an arbitrary collection of closed sets is not a closed set. 4+1=5

(b) If $\{a_n\}$ is a sequence of positive terms such that $\lim_{n \to \infty} a_n = l$, then prove that

$$\lim_{n \to \infty} (a_1 \, a_2 \dots a_n)^{\frac{1}{n}} = l$$

Using this result, show that

$$\lim_{n \to \infty} \frac{n+1}{(n!)^{\frac{1}{n}}} = e$$

3+2=5

(c) Show that the series

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \dots$$

is convergent. 5

(d) Test for convergence, absolute convergence and conditional convergence of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$$

5

(e) Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$. 5

(f) Test the differentiability of the function

$$f(x) = \begin{cases} x \tan^{-1}\left(\frac{1}{x}\right), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$$

at the point $x = 0$. 5

4. Answer *either* (a) and (b) *or* (c) and (d) : 5×2=10

(a) Prove that a bounded sequence with a unique limit is convergent. 5

(b) Using Sandwich theorem, show that the sequence $\{S_n\}$ where

$$S_n = \left\{ \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \right\}$$

converges to 1. 5

(c) Define a Cauchy sequence. Show that every Cauchy sequence is bounded. Give an example to prove that a bounded sequence need not be always a Cauchy sequence. 1+3+1=5

(d) Show that the sequence $\{S_n\}$ where

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{(n-1)!}$$

is convergent. 5

5. Answer *either (a)* and *(b) or (c)* and *(d)* : 5×2=10

(a) State Gauss' test for convergence of a series. Applying this test, examine the convergence of the series

$$1 + \frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \dots \infty$$

where $\alpha > 0$ and $\beta > 0$. 1+4=5

(b) Test the convergence of the series

$$1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \dots \infty, \text{ for } x > 0$$ 5

(c) Prove that every absolutely convergent series is convergent. Is the converse true? Justify. 3+2=5

(d) State Cauchy's root test for convergence of a series. Using this, test the convergence of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots\infty$$

1+4=5

6. Answer the following questions : 5×2=10

(a) Prove that if a function f is continuous on a closed interval $[a, b]$, then it attains its bounds at least once in $[a, b]$. 5

Or

Show that the function f defined on $\mathbb{R}$ by

$$f(x)=\begin{cases}1, & \text{if } x \text{ is rational}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$$

is discontinuous at every point.

(b) State Cauchy's criteria for existence of finite limit. Using this criteria, show that

$$\lim_{x\to 0}\frac{1}{x}\sin\frac{1}{x}$$

does not exist. 1+4=5

Or

Prove that a continuous and strictly increasing function f in $[a, b]$ is invertible and the inverse function is continuous in $[f(a), f(b)]$. 5

7. Answer any *two* parts : 5×2=10

(a) State and prove intermediate value theorem for derivatives. 1+4=5

(b) State Cauchy's mean value theorem. Using it, show that

$$\lim_{x\to 1} \frac{\cos\left(\frac{\pi x}{2}\right)}{\log\left(\frac{1}{x}\right)} = \frac{\pi}{2}$$ 1+4=5

(c) Evaluate the following limits : 2+3=5

(i) $\lim_{x\to 0} \frac{x - \log(1+x)}{1 - \cos x}$

(ii) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

(d) Find the maxima and minima of the function

$$f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x \quad \forall\, x \in [0, \pi]$$ 5

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