Total No. of printed pages = 8

3 (Sem 3) MAT M2

2015

MATHEMATICS

(Major)

Paper : 3.2

(Linear Algebra and Vector)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

GROUP – A

(Linear Algebra)

Marks : 40

1. Answer the following as directed : 1×7=7

(a) Show that in a vector space V(F)

$\alpha v = 0,\ v \neq 0 \Rightarrow \alpha = 0$

$\alpha v = 0,\ \alpha \neq 0 \Rightarrow v = 0$, where $v \in V$, $\alpha \in F$.

[Turn over

(b) Let S be the subset of $\mathbb{R}^3$ defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$$

Examine whether S is a subspace of $\mathbb{R}^3$.

(c) In $\mathbb{R}^3$, $\alpha = (4, 3, 5)$, $\beta = (0, 1, 3)$, $\gamma = (2, 1, 1)$.

Is α a linear combination of β and γ ?

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T(x_1, x_2) = (x_1, x_1 + x_2, x_2)$$

Examine whether T is a linear transformation.

(e) Let T be a linear operator on $\mathbb{R}^2$ which is represented by the following matrix :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with respect to the standard ordered basis. Then T has no eigen value in $\mathbb{R}$. –Justify whether it is true or false.

(f) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (x, 0)$. What is the eigen space of T associated with the eigen value 1 ?

(g) If λ is a simple eigen value (i.e. of multiplicity 1) of an $n \times n$ matrix A, then rank of $(A - \lambda T)$ is

(i) $n+1$

(ii) $n-1$

(iii) n

(iv) none of these

—Choose the correct option.

2. Answer the following questions : $2 \times 4 = 8$

(a) Let S and T be two non-empty finite subsets of a vector space v over a field F and $S \subset T$. Show that $L(S) \subset L(T)$, (where L(S) and L(T) denote the linear spans of S and T respectively).

(b) Let v and u be vector spaces over the same field F and $T : v \to u$ be a linear transformation. Show that $\ker T = \{0\}$ if and only if T is one-one.

(c) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T(x,y,z) = (3x - 2y + z,\ x - 3y - 2z)$.

Find the matrix of T relative to the ordered bases $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{R}^3$ and $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$.

(d) Using Cayley Hamilton theorem, compute the inverse of $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$.

3. Answer any *one* part : 5

(a) Find the range, rank, kernel and nullity of the following linear transformation :

$T : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$T(x, y) = (x + y, x - y, y)$.

(b) Determine the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors (1, 0, 0), (0, 1, 0), (0, 0, 1) of $\mathbb{R}^3$ to the vectors (1, 1), (2, 3) and (3, 2) respectively. Also determine ker T.

4. Answer the following questions : 10×2=20

(a) Let W be a subspace of a finite dimensional vector space V. Then show that W is also finite dimensional and $\dim W \leq \dim V$. Also show that $\dim V = \dim W$ if and only if V = W.

Or

Let W be a subspace of a finite dimensional vector space V. Prove that there exists a subspace W' of V such that $V = W \oplus W'$.

(b) Prove that similar matrices have same characteristic polynomial. Let A be a real n×n matrix. Let λ be a real eigen value of A. Show that there exists an eigen vector X of A corresponding to eigen value λ such that X is also real.

Or

Obtain the eigen values, eigen vectors and eigen spaces of the matrix :

$$A = \begin{pmatrix} 2 & 2 & -6 \\ 2 & -1 & -3 \\ 2 & -1 & 1 \end{pmatrix}$$

Is A diagonalisable ?

GROUP – B

(Vector)

Marks : 40

5. Answer the following as directed : 1×3=3

(a) If $\vec{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}$,

$\vec{b} = -5\hat{i} + 7\hat{j} - 3\hat{k}$,

$\vec{c} = 7\hat{i} - 5\hat{j} - 3\hat{k}$,

Find $\vec{a} \times (\vec{b} \times \vec{c})$.

(b) Examine whether $\vec{a}-2\vec{b}+3\vec{c})$, $-2\vec{a}+3\vec{b}-4\vec{c})$ and $\vec{a}-3\vec{b}+5\vec{c})$ are coplanar.

(c) $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$ is equal to

(i) $2\vec{b}$

(ii) $2\vec{a}$

(iii) 0

(iv) none of these

— Choose the correct option.

6. Show that if $\vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$, then

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2=\vec{a}^2\left(\vec{b}\times\vec{c}\right)^2$$ 2

7. Answer the following questions : 5×3=15

(a) Show that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ if and only if either $\vec{b}=0$, or $\vec{c}$ is collinear with $\vec{a}$, or $\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{c}$.

Or

If $\vec{a}=(1,1,1)$, $\vec{b}=(2,-1,3)$, $\vec{c}=(1,-1,0)$, $\vec{d}=(6,2,3)$, express $\vec{d}$ in terms of $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$.

(b) If $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + (a\tan\alpha)t\,\hat{k}$,

find $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|$

and $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]$

(c) Determine the constant 'a' so that the vector $\vec{f} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal.

8. Answer the following questions : 10×2=20

(a) A particle moves so that its position vector is given by $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$, where ω is a constant. Show that

(i) the velocity of the particle is perpendicular to $\vec{r}$

(ii) the acceleration is directed towards the origin and has magnitude proportional to the distance from the origin, and

(iii) $\vec{r} \times \frac{d\vec{r}}{dt}$ is a constant vector.

3+3+4=10

Or

If $\vec{a}$ is a constant vector, prove that

$$\operatorname{div}\left(r^n(\vec{a}\times\vec{r})\right)=0,$$

where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$. 10

(b) Evaluate : $\iint_S (y^2z^2\hat{i}+z^2x^2\hat{j}+x^2y^2\hat{k}).d\vec{S}$,

where S is the part of the sphere $x^2+y^2+z^2=1$ above the xy-plane. 10

Or

Find the work done when a force $\vec{F}=(x^2-y^2+x)\hat{i}-(2xy+y)\hat{j}$ moves a particle in xy-plane from (0, 0) to (1, 1) along the parabola $y^2 = x$. If C is the circle $x^2+y^2=4, z=0$, find the circulation of $\vec{F}$ along the curve C.

5+5=10