(c) If $\{x_n\}$ is a sequence, where $x_n = k\ (k \in R)$ is constant, then $\lim x_n = k$.

(Write true or false)

(d) The positive term series $\Sigma \frac{1}{n^p}$ is convergent if and only if

(i) $p > 1$

(ii) $p = 0$

(iii) $0 < p < 1$

(iv) $p \leq 1$

(Choose the correct answer)

(e) Define conditionally convergent of a series Σu_n.

(f) The Dirichlet's function f defined on R by

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is irrational} \\ -1 & \text{when } x \text{ is rational} \end{cases}$$

is

(i) continuous at $x = 1$

(ii) discontinuous at $x = 0$ only

(iii) continuous at every point of R

(iv) None of the above

(Choose the correct answer)

(g) Let a function f defined on the open interval $]0, 1[$ as follows :

$$f(x) = \frac{1}{x}, \quad \text{for all } x \in]0, 1[$$

Then f is

(i) continuous and uniformly continuous in $]0, 1[$

(ii) continuous but not uniformly continuous in $]0, 1[$

(iii) uniformly continuous but not continuous in $]0, 1[$

(iv) None of the above

(Choose the correct answer)

(h) Let f be a function defined by $f(x) = |x|$, $x \in [-1, 1]$. Write the condition on f for which Rolle's theorem is not applicable to f.

(i) Evaluate :

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x^2 - x}$$

(j) If a function f is

(i) continuous on $[a, b]$

(ii) derivable on $]a, b[$

(iii) $f'(x) > 0$, for all $x \in]a, b[$

then f is strictly _____ on $[a, b]$.

(Fill in the blank)

2017

MATHEMATICS

(Major)

Paper : 4.1

(Real Analysis)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×10=10

(a) Find the infimum of the set

$$\left\{\frac{(-1)^n}{n},\ n \in N\right\}$$

(b) What are the limit points of the sequence $\{x_n\}$, where

$$x_n = 2 + (-1)^n,\ n \in N\ ?$$

2. Answer the following questions : 2×5=10

(a) The open interval $]a, b[$ is neighbourhood of each of its points. Justify.

(b) Illustrate with an example that bounded sequence is not convergent.

(c) Show that the series $1+1+1+\cdots$ is not convergent.

(d) Examine the continuity at $x=2$ for the function $f(x)=x-[x]$, for all $x \geq 0$; $[x]$ denotes the largest integer $\leq x$.

(e) Find the value of $c \in]1, 2[$ of the Lagrange's mean value theorem, when

$$f(x)=2x^2+3x+4 \text{ in } [1, 2]$$

3. Answer any *four* parts : 5×4=20

(a) Prove that a set is closed if and only if its complement is open. 5

(b) Show that the sequence $\{x_n\}$, where

$$x_n = \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots + \frac{1}{\lfloor n}$$

is convergent. 5

(c) Test for convergence of the series

$$1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\cdots$$ 5

(d) Define alternating series. State the Leibniz's test for convergence of an alternating series. Applying the test, show that the series

$$\frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\cdots$$

is convergent. 1+1+3=5

(e) Show that the function

$$f(x)=|x|+|x-1|$$

is not derivable at $x=0$. 5

(f) Find the maximum and minimum values of y, where

$$y=(x-1)(x-2)^2$$ 5

4. Answer *either* (a) and (b) *or* (c) and (d) : 5×2=10

(a) If a and b be any two positive real numbers, then show that there exists a positive integer n such that $na > b$. 5

(b) State Sandwich theorem for sequence of real numbers. Applying this theorem, show that the sequence $\{x_n\}$, where

$$x_n=\left[\frac{1}{\sqrt{(n^2+1)}}+\frac{1}{\sqrt{(n^2+2)}}+\frac{1}{\sqrt{(n^2+3)}}+\cdots+\frac{1}{\sqrt{(n^2+n)}}\right]$$

converges to 1. 1+4=5

(c) Prove that the intersection of any finite number of open sets is open. Give an example to show that the intersection of arbitrary family of open sets is not open. 4+1=5

(d) Show that the sequence $\{x_n\}$ defined by recursion formula $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$ is monotonically increasing and bounded. Is the sequence convergent? 2+2+1=5

5. Answer *either* (a) and (b) *or* (c) and (d) : 5×2=10

(a) If $\sum_{n=1}^{\infty} u_n$ is a convergent series of positive terms and $u_{n+1} \le u_n$ for $n \in N$, prove that $\lim_{n \to \infty} (n u_n) = 0$. 5

(b) Using comparison test, show that the series $\Sigma\{\sqrt{(n^2+1)} - n\}$ is convergent. 5

(c) Applying logarithmic test, prove that the series

$$1 + \frac{x}{\lfloor 1} + \frac{2^2 x^2}{\lfloor 2} + \frac{3^3 x^3}{\lfloor 3} + \cdots$$

converges for $x < \frac{1}{e}$ and diverges for $x \ge \frac{1}{e}$. 5

(d) Prove that every absolutely convergent series is convergent. Is the converse true? Justify. 3+2=5

6. Answer any *two* parts : 5×2=10

(a) Prove that if a function is continuous in a closed interval, then it is bounded therein. 5

(b) Discuss the continuity of the following functions : 3+2=5

(i) $$f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}$$

at $x = 0$.

(ii) $$f(x) = \begin{cases} 3x+2, & 0 < x \leq 1 \\ \dfrac{x}{x-1}, & x > 1 \end{cases}$$

at $x = 1$.

(c) Define uniform continuity of a function on an interval. Show that the function f defined by $f(x) = x^3$, is uniformly continuous in $[-2, 2]$. 1+4=5

(d) If f is derivable at a and $f'(a) \neq 0$, then show that the function $1/f$ is also derivable at a and

$$\left(\frac{1}{f}\right)'(a) = \frac{-f'(a)}{\{f(a)\}^2}$$ 5

7. Answer any *two* parts : 5×2=10

(a) If a function f is

(i) continuous on $[a, b]$

(ii) derivable on $]a, b[$

then show that f is constant function on $[a, b]$ when $f'(x) = 0$, for all $x \in]a, b[$ and f is strictly decreasing on $[a, b]$ when $f'(x) < 0$, for all $x \in]a, b[$.

2+3=5

(b) State Cauchy's mean value theorem. Verify Cauchy's mean value theorem for the functions $f(x) = x^2$ and $g(x) = x^3$ in the interval $[1, 2]$. 1+4=5

(c) Show that

$$\frac{x}{1+x} < \log(1+x) < x, \quad \text{for all } x > 0$$

5

(d) Evaluate : 2+3=5

(i) $\lim_{x \to \pi/2} \frac{\log(x - \pi/2)}{\tan x}$

(ii) $\lim_{x \to 0} \left(\frac{1}{x} - \cot x \right)$

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